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Implications of Foundational Crisis in Mathematics: A Case Study in Interdisciplinary Legal Research

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IMPLICATIONS OF FOUNDATIONAL CRISES IN MATHEMATICS: A CASE STUDY IN INTERDISCIPLINARY LEGAL RESEARCH

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Abstract: As a result of a sequence of so-called foundational crises, mathematicians have come to realize that foundational inquiries are difficult and perhaps never ending. Accounts of the last of these crises have appeared with increasing frequency in the legal literature, and one piece of this Article examines these invocations with a critical eye. The other piece introduces a framework for thinking about law as a discipline. On the one hand, the disciplinary framework helps explain how esoteric mathematical topics made their way into the legal literature. On the other hand, the mathematics can be used to examine some aspects of interdisciplinary legal research.

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I. INTRODUCTION

> *For whatever may be said about the importance of aiming at depth rather than width in our studies, and however the demand of the present age may be for specialists, there will always be work, not only for those who build up particular sciences and write monographs on them, but for those who open up such communications between the different groups of builders as will facilitate a healthy interaction between them.*
>
> *James Clerk Maxwell*[1]

This Article consists of two related pieces. One piece examines invocations by legal scholars of what mathematicians[2] describe as "the *third foundational crisis* that mathematics is still undergoing."[3] Casual readers of law reviews might be astonished to discover that accounts of various aspects of this crisis have appeared with increasing frequency in the legal literature,[4] and one piece of this Article examines these invocations with a critical eye. The other piece introduces a framework for thinking about law as a discipline. Central to this framework is a particular conception of the Western intellectual tradition in terms of disciplines. The notion is that a discipline is at once a science, an art, and a technology. How are these seemingly disparate pieces related in this "case study"? On the one hand, the disciplinary framework helps explain why esoteric mathematical topics such as Gödel's Theorems and non-Euclidean geometry have appeared in the legal literature. On the other hand, the mathematics is used to examine some aspects of interdisciplinary legal research. The rest of this Introduction expands the sketch of the Article presented thus far.

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1. *Quoted in* Theodore M. Porter, *The Rise of Statistical Thinking 1820–1900*, at 230 (1986).

2. For the purposes of this Article, no distinction will be made between mathematicians, mathematical logicians, and philosophers of mathematics.

3. Abraham Fraenkel et al., *Foundations of Set Theory* 14 (2d rev. ed. 1984).

4. *See* M.B.W. Sinclair, *Evolution in Law: Second Thoughts*, 71 U. Det. Mercy L. Rev. 31, 33 n.13 (1993). Indeed, there are three recent pieces devoted largely to various aspects of the crisis. *See* Mark R. Brown & Andrew C. Greenberg, *On Formally Undecidable Propositions of Law: Legal Indeterminacy and the Implications of Metamathematics*, 43 Hastings L.J. 1439 (1992); David R. Dow, *Gödel and Langdell—A Reply to Brown and Greenberg's Use of Mathematics in Legal Theory*, 44 Hastings L.J. 707 (1993); John M. Rogers & Robert E. Molzon, *Some Lessons About the Law from Self-Referential Problems in Mathematics*, 90 Mich. L. Rev. 992 (1992).

As a result of a sequence of so-called foundational crises, mathematicians have come to realize that foundational inquiries are difficult and perhaps never ending. Although accounts of this crisis have appeared with increasing frequency in the legal literature, there are a number of problems with these presentations.

At the most basic level, there are problems with attempts merely to state the mathematics. One law review article tells us that "as Kurt Gödel demonstrated, any formal logical system ultimately rests on some undecidable—that is, unprovable—propositions."[5] Another refers to "Gödel's proof of ultimate inconsistency in mathematics."[6] A third explains that "a 'complete' theorem is inconsistent if it is an axiom. Nothing purely complete is proved."[7] Think this sounds like gobbledygook? It is. At a minimum, this Article provides the reader with an understanding of the mathematics involved.

There also are problems with attempts to apply the mathematics to law. Many authors, for example, use mathematics to bolster or attack various positions on "legal indeterminacy."[8] One legal scholar tells us flatly that "[t]he implications of Gödel's Theorems for any theory of law have been ignored for too long . . . . Every theory of law is incomplete."[9] Think that it can't be this simple? It isn't. Authors, for example, routinely ignore the important fact that results such as Gödel's Theorems only apply in a very specific setting. This Article does not evaluate the ultimate conclusions reached through these types of invocations. The Article does ask, however, whether scholars have carefully considered the mathematics they invoke.

Perhaps most troubling are uses of the current mathematical crisis as support for a general intellectual skepticism. Indeed, scholars such as Edward Purcell and Joan Williams have traced the intellectual roots of current critiques of the Western intellectual tradition, including critical legal studies, in large part to the current mathematical crisis.[10] One legal commentator tells us that "it should not be surprising to find that the philosophical implication of Gödel's theorem should question the basic

5. Lee Loevinger, *Standards of Proof in Science and Law*, 32 Jurimetrics 323, 343 (1992).

6. Keith Aoki, *Contradiction and Context in American Copyright Law*, 9 Cardozo Arts & Ent. L.J. 303, 382 (1991) (quoting Robert Venturi, *Complexity and Contradiction in Architecture* 16 (1966)).

7. Leonard R. Jaffee, *Empathic Adjustment—An Alternative to Rules, Policies, and Politics*, 58 U. Cin. L. Rev. 1161, 1193 (1990).

8. *See infra* text accompanying notes 345–47.

9. Daniel Kornstein, *The Music of the Laws* 127 (1982).

10. *See infra* text accompanying notes 376–79 and 389–91.

premise of philosophy—that is, the basic question of whether reality exists."[11] Think there must be more to the story? There is. If only used metaphorically, such invocations may not be problematic in the same sense as previously described. Yet a mathematician still can point out why a law review article assertion that the rejection of objective truth and certainty has "permeated . . . mathematics"[12] is misleading, if not inaccurate. The mathematician also can note that as a result of its crises mathematics has matured, not declined, as a discipline.

Obviously, these problems are related. The first type leads to the second, which in turn leads to the third. In any case, they stem from a lack of attention to the intellectual history of the current crisis. Accordingly, the mathematical piece of this Article devotes a large amount of space to presenting this history.

The second piece of this Article stems from the assertion that in an era in which higher education in general and legal education in particular are undergoing something of an identity crisis, a starting point for an examination of law's place in the modern university setting is essential. For the purposes of this Article, the starting point is the notion of a discipline. The premise of this notion of discipline is that the basic goal underlying the Western intellectual tradition is to understand, appreciate, and utilize our environment. Understanding (i.e. science) involves classification, appreciation (i.e. art) involves interpretation, and utilization (i.e. technology) involves the means for providing sustenance and comfort. The environment, however, is complex and textured, and thus the intellectual tradition centers on disciplines—more focused approaches to the basic goal. A discipline is at once a science, an art, and a technology. A discipline is characterized as a science by the objects considered, the properties studied, and the classification employed. As an art, a discipline is characterized by the range of interpretations given and the symbolic medium used. Finally, a discipline is characterized as a technology by the methods and scope of its applications. Within each discipline, science, art, and technology work together to present a specific part of a world view.

To assert that law is a discipline is to assert that law is a science, that law is an art, and that law is a technology. The core of understanding (i.e. science) is classification. There is no attempt in this Article to provide

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11. Randall Kelso, Book Note, 1981 Wis. L. Rev. 822, 833 n.44 (reviewing Douglas R. Hofstadter, *Gödel, Escher, Bach: An Eternal Golden Braid* (1980)).

12. Joan C. Williams, *Critical Legal Studies: The Death of Transcendence and the Rise of the New Langdells*, 62 N.Y.U. L. Rev. 429, 430–31 (1987).

any classification scheme for law. The Article merely asserts that classification is the heart of the scientific component of law as a discipline. That law is a technology is probably the least controversial of the assertions, while the assertion that law is an art perhaps is the most appealing yet the hardest to accept. The resulting legal world view is not only what makes it possible to write a book on comparative law, but also what makes it easier for a lawyer than a mathematician to digest it.

This Article focuses on law as a science. For example, some time is spent discussing Christopher Columbus Langdell's controversial views on the matter. In fact, the disciplinary part of the Article is the most important, and future case study articles will discuss other aspects of law as a discipline. Thus, this Article is the first part of a work in progress.

What is the relation between the seemingly disparate mathematical and disciplinary pieces described above? On the one hand, the disciplinary framework helps explain how esoteric mathematical topics such as Gödel's Theorems and non-Euclidean geometry made their way into the legal literature. The past 150 years have seen a complex and comprehensive reevaluation of the Western intellectual tradition. Indeed, recent intellectual critiques such as Marxism and post-modernism can be characterized in terms of fundamental reevaluations of science, art, and technology. From this perspective, the use of the current mathematical crisis, both within and outside of law, represents a particular manifestation of the overall reevaluation of science. On the other hand, the Article uses specific critiques of the invocations of the current crisis to make some general points about meaningful interdisciplinary research. Few doubt Maxwell's assertion that interdisciplinary research is important.[13] Such research sharpens the resolution of the intellectual picture by applying the perspectives of differing disciplines to the most interesting aspects of the environment. Moreover, interdisciplinary work has been, and continues to be, a major force in the creation and evolution of disciplines themselves. A major intellectual challenge today, however, is fostering such work in an information-rich era in which it is difficult to master even a small part of a given field. Much of the discussion in the legal literature of the current foundational crisis in mathematics illustrates two basic difficulties in doing meaningful interdisciplinary legal research: gaining a sufficient understanding of what is often a foreign discipline, and employing that discipline in a manner that reflects both its relevance to, and separateness from, law.

13. *See supra* text accompanying note 1.

Admittedly, there is much to be said for using something other than invocations of the current foundational crisis as a case study. Although the mathematics is interesting *per se* and involves some of the crowning achievements of modern thought, it is subtle and somewhat difficult. Nevertheless, legal scholars have put it in play and, as indicated above, in a big way. The only responsible response is to face the mathematics head on. Moreover, addressing the mathematics does illustrate the magnitude of the effort that often is required for meaningful interdisciplinary research.

Part II of this Article presents more detail on the disciplinary framework. The reader should keep in mind, however, that this framework will be more fully developed and illustrated in future articles. Part III contains a discussion of law as a science, focusing on Langdell's views on the matter. Parts II and III set the stage for part IV's discussion of the legal invocations of the current mathematical crisis. After some preliminary comments in section A of part IV, section B provides a careful description of the content and context of the foundational crises. Section B also provides a detailed analysis of the accuracy of descriptions in the legal literature. Section C uses the discussion presented in section B to examine some of the invocations of the mathematical material. Part V presents a few final observations. Some patience is required for section B of part IV. The mathematics is important for developing the disciplinary points of the Article. Moreover, any meaningful use in law of the current foundational crisis must begin with a careful study of the mathematics.

II. DISCIPLINES AND THE WESTERN INTELLECTUAL TRADITION

> *Man is a singular creature. He has a set of gifts which make him unique among the animals: so that, unlike them, he is not a figure in the landscape—he is a shaper of the landscape. In body and in mind he is the explorer of nature . . . .*
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> . . . .
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> *. . . Man is distinguished from other animals by his imaginative gifts . . . [so that] the great discoveries of different ages and different cultures, in technique, in science, in the arts, express in their progression a richer and more intricate conjunction of human faculties, an ascending trellis of his gifts.*
>
> *Jacob Bronowski*[14]

A. *The Notion of a Discipline*

The premise for this Article's notion of a discipline is that the basic goal underlying the Western intellectual tradition is to understand, appreciate, and utilize our environment.[15]

Understanding refers to science. This use of the word "science" connotes systemization and organization, as opposed to its more narrow association with what usually are called the natural sciences.[16] Science involves the classification of the objects appearing in the environment according to their important properties.[17] These objects may be sensory or non-sensory, and the exact nature of the classification, such as description, prediction, prescription, or explanation, depends on the context.[18] This admittedly is an older use of the word "science,"[19] but it

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14. Jacob Bronowski, *The Ascent of Man* 19–20 (1973).

15. Nothing said here is meant to imply that other traditions lack these concepts.

16. For a discussion of this distinction, see David A. Funk, *Juridical Science Paradigms As Newer Rhetorics in 21st Century Jurisprudence*, 12 N. Ky. L. Rev. 419, 435 (1985).

17. *See Webster's Seventh New Collegiate Dictionary* 771 (1972) (giving as one definition "a department of systematized knowledge as an object of study <the [science] of theology>").

For the purposes of this Article, I am not interested in debates about the ontology of the objects "appearing" in the environment.

18. A general discussion of classification is provided in Stephan Körner, *Classification Theory*, in 4 *Encyclopaedia Britannica* 691 (15th ed. 1974). For more detailed discussions in specific contexts, see Richard B. Braithwaite, *Scientific Explanation: A Study of the Function of Theory, Probability*

captures the essence of one of the three basic dimensions of the intellectual tradition.

Appreciation refers to art. This use of the word "art" connotes aesthetics, as opposed to its more narrow association with what usually are called the fine arts. Art involves the interpretation of the environment through the creative use of a symbolic medium.[20] As with the concept of science, this Article does not attempt to fill out these ideas completely. In particular, no effort is made to develop the concepts of the artist, the work of art, and the spectator.[21] Nonetheless, as with science and technology, the intent here is to present notions that cut across disciplines.[22]

Finally, utilization refers to technology. This use of the word "technology" is intended to connote application, as opposed to its more narrow association with what usually are called the engineering sciences. Technology involves the means employed to provide sustenance and comfort.[23]

The environment is complex and textured, and thus the intellectual tradition centers on disciplines—more focused approaches to the basic goal described above. A discipline is at once a science, an art, and a

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and Law in Science 9–10 (1953) (discussing scientific laws); Carl G. Hempel, *Aspects of Scientific Explanation and Other Essays in the Philosophy of Science* 135–73 (1965) (discussing natural and social sciences); W.S. Jevons, *The Principles of Science* 673–734 (2d ed. 1877) (discussing natural sciences); Thomas Munro, *The Arts and Their Interrelations* (2d ed. 1967) (discussing arts).

19. Some even view this use as "obsolete." *See* Barbara J. Shapiro, *Law and Science in Seventeenth-Century England*, 21 Stan. L. Rev. 727, 727 (1969).

20. *See* John A. Fisher, *Reflecting on Art* 324 (1993) ("Four major concepts dominate traditional aesthetics: beauty, representation, expression, and form."); Dewitt H. Parker, *Aesthetics*, in *Encyclopedia of the Arts* 14, 15 (Dagobert D. Runes & Harry G. Schrickel eds., 1946) ("Three general conceptions of art have dominated the history of aesthetics: *imitation*, *imagination*, and *expression* or *language*.").

21. Any such effort would move far beyond the scope of this Article and into controversial areas. *See* Hugh Curtler, *What Is Art?* 1, 3 (1983).

22. *Cf.* Helen Gardner, *Understanding the Arts* 318–28 (1932).

No attempt is made here to define the term "art" as it is used in connection with music, painting, etc. Indeed, there is perhaps nothing more problematic. *See* Horst W. Janson, *History of Art* 9 (1967). For a discussion of the problems involved, see Paul Ziff, *The Task of Defining a Work of Art*, 62 Phil. Rev. 58 (1953). For a collection of standard perspectives, see Frank A. Tillman & Steven M. Cahn, *Philosophy of Art and Aesthetics from Plato to Wittgenstein* (1969). For general overviews, see Monroe C. Beardsley, *Aesthetics from Classical Greece to the Present: A Short History* (1966); Munro, *supra* note 18, at 49–109.

23. *See Webster's Seventh New Collegiate Dictionary* 905 (1972) (defining technology as "the totality of the means employed to provide . . . human sustenance and comfort").

technology.[24] A discipline is characterized as a science by the objects considered, the properties studied, and the classification employed. The expression of this characterization is the corpus or set of significant statements of the field. As an art, a discipline is characterized by the range of interpretations and the symbolic medium used. Finally, a discipline is characterized as a technology by the methods and scope of its applications. Within each discipline, science, art, and technology work together[25] to present a specific part of a world view.[26]

B. *Mathematics As a Discipline*

To make the concepts introduced in the last section more concrete, consider mathematics as a science, an art, and a technology.[27] Mathematics is used here rather than law for two reasons. First, one part of this Article deals with subtle mathematical concepts, and therefore the Article introduces some mathematics here. Second, the use of mathematics makes it possible to illustrate the concepts of science, art, and technology in a context in which they are not likely to be controversial. Part III of the Article discusses some of these notions in the context of law.

As a science, mathematics deals with objects whose essential properties involve number, shape, and function.[28] Indeed, these three

24. For another "dimensional" approach to the notion of a discipline, see Timothy P. Terrell, *Flatlaw: An Essay on the Dimensions of Legal Reasoning and the Development of Fundamental Normative Principles*, 72 Cal. L. Rev. 288 (1984).

25. Others posit an antagonistic relationship. *Cf.* Daniel Bell, *The Coming of Post-Industrial Society* 374–77 (1973) (discussing potential antagonism between "scientific, technological, and cultural estates"); *The Republic of Plato* b.X (Francis M. Cornsford trans., 1941) (discussing potential corrupting influence of fine arts on search for knowledge).

26. What does it mean, after all, to say that students are taught to think like a lawyer? The notion of the separation of disciplines can be traced back at least as far as Aristotle. *See* T.Z. Lavine, *From Socrates to Sartre: The Philosophic Quest* 76 (1984). The current scope and organization of disciplines has been affected in part by the social and institutional factors that accompanied the transition from the educated amateur, to the professional society, to the modern research university. *See* Roger L. Geiger, *To Advance Knowledge: The Growth of American Research Universities 1900–1940*, at 20–27 (1986). For discussions of the American version of this transition, see *The Organization of Knowledge in Modern America 1860–1920* (Alexandra Oleson & John Voss eds., 1979) [hereinafter Oleson & Voss]; *The Pursuit of Knowledge in the Early American Republic: American Scientific and Learned Societies from Colonial Times to the Civil War* (Alexandra Oleson & Sanborn C. Brown eds., 1976).

27. *See* Morris Kline, *Mathematics: A Cultural Approach* 1–10 (1962).

28. *See* Nat'l Research Council Bd. on Mathematical Sciences, *Mathematical Sciences: Some Research Trends* 21 (1988) [hereinafter *Research Trends*]. Of the three, the notion of function may be the least familiar to the general reader. Roughly speaking, a function from an input set A to a set B

concepts provide the starting points for the traditional branches of algebra, geometry, and analysis.[29] The heart of mathematical classification is the deductive method.[30] Using this method, mathematics generates its corpus, a typical example of which is the statement that there is no rational number whose square is two.[31] As an art, mathematics interprets using number, shape, and function in conjunction with a

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consists of the assignment to each element of A one and only one element of B. For example, if A is the set of natural numbers {0, 1, 2, . . .} and B also is the set of natural numbers, then the assignment of a number to its square is a function. This squaring function might be denoted by a single letter and described in terms of the value assigned to an arbitrary member of its input set. That is, the squaring function might be denoted by f and described by saying that $f(n) = n*n$. For a brief introduction to the notion of function. see 1 Tom M. Apostol, *Calculus* 50–54 (2d ed. 1967).

29. *See* Morris Kline, *Mathematical Thought from Ancient to Modern Times* 949–54 (1972) [hereinafter Kline, *Mathematical*] (discussing role of functions in development of analysis); Morris Kline, *Geometry*, Sci. Am., Sept. 1964, at 60 (discussing role of shape in development of geometry); W.W. Sawyer, *Algebra*, Sci. Am., Sept. 1964, at 70 (discussing role of number in development of algebra).

Obviously, a complete description of these branches would amount to a substantial mathematical education. The Sawyer and Kline articles present elementary introductions to algebra and geometry. For an elementary introduction to analysis. see Edward Kasner & James R. Newman, *Mathematics and the Imagination* 299–356 (1940). The traditional branches are replete with interactions and subdivisions, and they have been supplemented by a variety of new areas. For a general introduction, see 1, 2 Edna E. Kramer, *The Nature and Growth of Modern Mathematics* (1970).

30. *See* Yu I. Manin, *A Course in Mathematical Logic* 48 (1977) ("[T]he ideal for what constitutes a mathematical demonstration of a 'nonobvious truth' has remained unchanged since the time of Euclid: we must arrive at such a truth from 'obvious' hypotheses, or assertions which have already been proved, by means of a series of explicitly described, 'obviously valid' elementary deductions.").

This is the "ideal," but the required rigor has changed from time to time. *See* Raymond L. Wilder, *Relativity of Standards of Mathematical Rigor*, in 3 *Dictionary of the History of Ideas* 170 (1973); *see also* Judith V. Grabiner, *Is Mathematical Truth Time-Dependent?*, 81 Am. Mathematical Monthly 354 (1974).

31. Recall that a rational number can be expressed as a fraction p/q where p and q are integers 0, 1, -1, 2, -2, . . . , and q is not zero. Recall also that a fraction can be reduced to lowest terms by removing common factors. For example, the reduced form of 16/10 is 8/5, obtained by removing the common factor 2.

Now suppose to the contrary that there is a rational number r such that $r*r = 2$. Suppose $r = p/q$ in lowest terms. By the assumption on r, we have that $(p/q)*(p/q) = 2$. By simple algebra, this implies that $p*p = 2q*q$. But $2q*q$ is an even number, so that p must be even since an odd number times an odd number is odd. Say $p = 2k$. Substituting $2k$ for p in the equality $p*p = 2q*q$, one obtains $q*q = 2k*k$, implying that q is even. Because p and q are even, they have a common factor of 2. One concludes from this contradiction that there is no rational number whose square is two.

G.H. Hardy called this a "mathematical theorem[] . . . which every mathematician will admit to be[ing] first rate[,] . . . simple both in idea and in execution, but there is no doubt at all about [its] being [a theorem] of the highest class[,] . . . as fresh and significant as when it was first discovered—two thousand years have not written a wrinkle on [it]." G.H. Hardy, *A Mathematician's Apology* 91–92 (3d prtg. 1967).

This result actually is relevant to this Article! *See infra* text accompanying note 104.

medium that consists of a highly refined symbolic language.[32] Euclidean geometry, for example, represents an elegant interpretation of some of the spatial aspects of the environment. Finally, mathematics as a technology is used to solve problems that can be modeled in terms of number, shape, and function.[33]

The three dimensions of mathematics interact symbiotically.[34] The history of mathematics contains many examples in which mathematics as a science has been driven by mathematics as a technology and *vice-versa*. For example, the development of the field of probability is due largely to the investigation of a number of practical problems.[35] Conversely, the study of prime numbers,[36] long considered to be of little practical use,[37] has become immensely important in a variety of applied areas.[38] There is a similar relationship between mathematics as a science and mathematics as an art. The symbolic language of mathematics helps guide mathematical reasoning.[39] Conversely, the development of non-

32. *See* Henri Poincaré, *The Relations of Analysis and Mathematical Physics*, 4 Bull. Am. Mathematical Soc'y 247, 248 (1898) ("[Mathematics has] an end esthetic. . . . [A]depts find in mathematics delights analogous to those that painting and music give. They admire the delicate harmony of number and of forms; they are amazed when a new discovery discloses for them an unlooked for perspective . . . ."). For other discussions of mathematics as an art, see Nathan A. Court, *Mathematics in Fun and Earnest* 127–40 (1964); P.R. Halmos, *Mathematics As a Creative Art*, 56 Am. Scientist 375 (1968); J.W.N. Sullivan, *Mathematics As an Art*, in *Aspects of Science: Second Series* 80 (1926); Henri Poincaré, *Mathematical Creation*, Sci. Am., Aug. 1948, at 54. *See also* Scott Buchannan, *Poetry and Mathematics* (1929); Jerry P. King, *The Art of Mathematics* (1992).

Many mathematicians exalt this dimension of mathematics above all others. *See* Lynn A. Steen, *Mathematics Today*, in *Mathematics Today* 1, 10 (Lynn A. Steen ed., 1978) ("[B]eauty and elegance have more to do with the value of a mathematical idea than does either strict truth or possible utility."); *see also* Hardy, *supra* note 31.

33. For a general discussion with a number of interesting examples, see Felix E. Browder & Saunders MacLane, *The Relevance of Mathematics*, in *Mathematics Today* 323 (Lynn A. Steen ed., 1978).

34. The technological dimension at times has existed in an uneasy alliance with the other two. *See* Philip J. Davis & Reuben Hersh, *The Mathematical Experience* 85–89 (1981); *Research Trends*, *supra* note 28, at 2–3.

35. *See* Ian Hacking, *The Emergence of Probability* 11–12 (1984).

36. Recall that a natural number 0, 1, 2, . . . is prime if it is larger than 1 and divisible only by 1 and itself. Thus, 7 is prime, but 0, 1, and 9 are not.

37. As late as 1940, Hardy could write that "[we] may be justified in rejoicing that there is one science at any rate [number theory] . . . whose very remoteness from ordinary human activities should keep it gentle and clean." Hardy, *supra* note 31, at 121.

38. For a general discussion with a number of examples, see M.R. Schroeder, *Number Theory in Science and Communication* (2d enlarged ed. 1986).

39. *See* G. Polya, *How to Solve It* 134–41 (2d ed. 1957).

Euclidean geometries illuminated the nature of mathematics as an art.[40] Finally, there is a symbiotic relationship between mathematics as an art and mathematics as a technology. Indeed, the symbolic medium of mathematics largely defines the scope of its applications.[41] Conversely, practical problems involving the motion of objects in a plane led to the general study of curves in the plane,[42] many of which are known for their elegance and beauty.[43]

As a science, an art, and a technology, mathematics presents a unique perspective on the environment. A fundamental reason for learning mathematics is to be exposed to what it means to view the world like a mathematician.[44]

C. *Interdisciplinary Research*

Disciplines share certain commonalities because the most interesting aspects of the environment appear in many different guises. The resulting interactions manifest themselves in at least two ways. First, disciplines often are categorized by a common emphasis or approach to one or more of the three basic dimensions.[45] There is, for example, the rough division of disciplines into the natural sciences, the social sciences, and the humanities. Second, and more importantly, disciplines intersect. To consider just one example, the study of DNA fingerprinting illustrates how a wide-ranging collection of disciplines (law, genetics, mathematics, etc.) deals with the concept of coincidence.[46]

Interdisciplinary research sharpens the resolution of the intellectual picture of the environment by applying different perspectives to the most interesting aspects of the environment. Moreover, such research is a

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40. *See* Sullivan, *supra* note 32.

41. For many examples illustrating this principle, see M.M. Schiffer & Leon Bowden, *The Role of Mathematics in Science* (1984).

42. *See* Kline, *Mathematical, supra* note 29, at 544–54.

43. The reader is encouraged to look through the figures in J. Dennis Lawrence, *A Catalog of Special Plane Curves* (1972).

44. *See* Kline, *Mathematical, supra* note 29, at 1–10.

45. This classification of disciplines, each of which involves classifications, raises the question of a metastance. A full discussion of this question is beyond the scope of this Article.

For a legally-based introduction to the "meta," see Stuart Banner, *Please Don't Read the Title*, 50 Ohio St. L.J. 243 (1989). In any case, "groundedness" problems are nothing new. *See, e.g.*, 1 Frederick Copleston, *A History of Philosophy* 295 (1946) (discussing Aristotle's critique of Plato's theory of universals).

46. For an overview, see Comm. on DNA Technology in Forensic Sciences, Nat'l Research Council, *DNA Technology in Forensic Science* (1992).

major force in the creation and evolution of disciplines themselves. Scholars, however, must be aware of two basic difficulties in doing meaningful interdisciplinary research: gaining a sufficient understanding of what is often a foreign discipline,[47] and employing that discipline in a manner that reflects both its relevance and separateness.[48] The major intellectual challenge today is fostering meaningful interdisciplinary work in an information-rich era in which it is difficult to master even a small part of a given field.

D. *Reevaluating the Western Intellectual Tradition*

The past 150 years have seen a complex and comprehensive reevaluation of the Western intellectual tradition.[49] In its negative sense, the reevaluation involves a fundamental reexamination of the ideas of science, art, and technology.[50] In its positive sense, the reevaluation encompasses a variety of attitudes ranging from evolution to revolution to anarchy. This Article is more concerned with the negative sense. With respect to science, foundational crises challenge the traditional bases of classification schemes.[51] With respect to art, certain theories question the notions of creativity, interpretation, and symbolism.[52] With respect to technology, critics expose and probe normative starting points. The archetype of the technological component of the reevaluation is Karl

47. *See* Brian Leiter, *Intellectual Voyeurism in Legal Scholarship*, 4 Yale J.L. & Human. 79, 79–80 (1992); Sinclair, *supra* note 4, at 32; Mark Tushnet, *Critical Legal Studies: A Political History*, 100 Yale L.J. 1515, 1515 n.1 (1991); Mark Tushnet, *Truth, Justice, and the American Way: An Interpretation of Public Law Scholarship in the Seventies*, 57 Tex. L. Rev. 1307, 1338 n.140 (1979).

48. *See* Charles W. Collier, *Interdisciplinary Legal Scholarship in Search of a Paradigm*, 42 Duke L.J. 840 (1993); Charles W. Collier, *The Use and Abuse of Humanistic Theory in Law: Reexamining the Assumptions of Interdisciplinary Legal Scholarship*, 41 Duke L.J. 191 (1991); Dow, *supra* note 4, at 724; Robin L. West, *Adjudication Is Not Interpretation: Some Reservations About the Law-As-Literature Movement*, 54 Tenn. L. Rev. 203 (1987).

49. *See* Pauline M. Rosenau, *Post-Modernism and the Social Sciences* 4 (1992) ("[A] radically new and different . . . movement is coalescing in a broad-gauged re-conceptualization of how we experience and explain the world around us."); *see also* Günter Frankenberg, *Down by Law: Irony, Seriousness, and Reason*, 83 Nw. U. L. Rev. 360, 371 (1989) ("[A]n epochal intellectual-political battle has been raging . . . .").

50. *See* Rosenau, *supra* note 49, at 15–17.

51. Commentators have denoted this challenge in a variety of ways. *See* Edward A. Purcell, Jr., *The Crisis of Democratic Theory: Scientific Naturalism and the Problem of Value* 47–73 (1973) (using phrase "non-Euclideanism"); Williams, *supra* note 12, at 429–39 (using phrase "new epistemology").

52. For an introduction to some of these theories, see Terry Eagleton, *Literary Theory: An Introduction* (1983).

Marx's economic analysis of class conflict.[53] His technological focus was symptomatic of the aftermath of the so-called Revolutions of 1848,[54] and this date is the somewhat arbitrary starting point for the 150 year period mentioned above.

The reevaluation consists of discipline-centered critical schools, such as the critical legal studies movement, and discipline-independent critical stances, such as Marxism, feminism, and post-modernism.[55] The reevaluation might be viewed in terms of a three-dimensional "critical space." One axis includes the critical stances, and another axis includes the critical schools. That is, Marxism cuts across a variety of disciplines, and the critical legal studies movement embraces a variety of critical stances. A third axis consisting of science, art, and technology represents the scope of a particular challenge. A given critic will occupy a region of this three dimensional space.

This description, however, hides a number of complexities. Critical stances or schools may be subdivided. Moreover, although there is much agreement about questioning the intellectual *status quo*, there are many tensions hidden beneath this common cause.[56] Finally, one commentator notes that any attempt at a description "may be inherently objectionable to those . . . who view such endeavors as necessarily misguided, as flawed attempts at systemization."[57] Such a description, however, does indicate that one aspect of the reevaluation consists of engaging the tradition on the tradition's own terms. In this sense, the current reevaluation can be viewed as the most recent incarnation of a dialectic that has characterized Western thought since its inception.[58]

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53. For a brief introduction to his ideas, see Lavine, *supra* note 26, at 288–301.

54. *See* Charles Breuning, *The Age of Revolution and Reaction, 1789–1850*, at 276–78 (2d ed. 1977).

55. Some of these stances, such as Marxism, began within a particular discipline. For a brief introduction to the development of this stance, see Lavine, *supra* note 26, at 261–320.

56. *See* Rosenau, *supra* note 49, at 14 (discussing tension within post-modernism); *id.* at 6, 158–60 (discussing tension between Marxism and post-modernism).

57. *Id.* at 19.

58. Modern epistemological arguments, for example, can be traced back to ancient Greek disputes. *See* Mary Tiles & Jim Tiles, *An Introduction to Historical Epistemology: The Authority of Knowledge* 52–53 (1993). Moreover, classical Greek literary criticism encompassed a variety of positions, many of which anticipated modern issues and stances. *See* 1 *Cambridge History of Literary Criticism* at x-xi, 346 (George A. Kennedy ed., 1989). Finally, Athenians engaged in a robust debate about the normative premises underlying their society. *See* Victor Ehrenburg, *From Solon to Socrates: Greek History During the Sixth and Fifth Centuries B.C.* 48–74 (1968); Ivan M. Linforth, *Solon the Athenian* 46–91 (1919).

III. IS LAW A DISCIPLINE?

Langdell seems to have been an essentially stupid man who, early in his life, hit on one great idea to which, thereafter, he clung with all the tenacity of genius. Langdell's idea evidently corresponded to the felt necessities of the time. However absurd, however mischievous, however deeply rooted in error it may have been, Langdell's idea shaped our legal thinking for fifty years.

Langdell's idea was that law is a science.

Grant Gilmore[59]

A. *Introduction*

The framework of this Article presupposes that law is a discipline. This part of the Article explores one aspect of that supposition—that law is a science (i.e. that law involves classification).[60] Any discussion of law as a science must deal with the controversy surrounding Langdell's conception of the relationship between law and science.[61] This part examines two components of his views: that the study of law is suitable for the modern research university and that law involves science.

B. *The Emergence of the American Law School As Part of the Modern American Research University*

The emergence of the American research university is a complex phenomenon.[62] The research university can be traced in part to three

59. Grant Gilmore, *The Ages of American Law* 42 (1977).

60. It will be assumed that law has technological and artistic components. The assertion that law is a technology should not be controversial. For one discussion of law as an art, see Laura S. Fitzgerald, Note, *Towards a Modern Art of Law*, 96 Yale L.J. 2051 (1987). For a brief discussion of the tensions between the scientific and technological components as they manifest themselves in legal education, see Carrie Menkel-Meadow, *Narrowing the Gap by Narrowing the Field: What's Missing from the MacCrate Report—Of Skills, Legal Science, and Being a Human Being*, 69 Wash. L. Rev. 593, 596–603 (1994).

61. There is some question whether Langdell or Harvard President Charles Eliot should be considered the more important figure. *See* John H. Schlegel, *American Legal Theory and American Legal Education: A Snake Swallowing It's* [sic] *Tail?*, in *Critical Legal Thought: An American-German Debate* 49, 51 (Christian Joerges & David M. Trubek eds., 1989); *see also* Robert Stevens, *Law School: Legal Education in America from the 1850s to the 1980s* 36 (1983).

62. For general discussions, see John S. Brubacher & Willis Rudy, *Higher Education in Transition: A History of American Colleges and Universities 1636–1976* (3d ed. 1976); Frederick

specific reforms proposed for American higher education.[63] First, institutions should give greater scope to new disciplines, particularly the natural sciences. Second, they should offer training preparing students for specific careers. Third, they should reflect educational trends in European countries, particularly Germany with its emphasis on faculty-oriented advanced study and research. The research university also can be traced to the widespread institutional growth that characterized the post-Civil War United States.[64] Like the mammoth networks that came to dominate manufacturing and energy, the research university system gained control over the production and diffusion of basic knowledge.[65] As a result of these two types of influences, higher education moved beyond the pre-Civil War model.[66] Private schools such as Harvard, state schools such as Michigan, and entirely new institutions such as Johns Hopkins sought a new ideal.[67]

American legal education faced a similar pair of influences. As with higher education in general, law schools were examining the need for educational reform.[68] Just prior to Langdell's arrival, the Harvard Law School could be described in the following terms:

> [It was] a school without examination . . . or degree. [It] had a faculty of three professors giving but ten lectures a week to one hundred and fifteen students of whom fifty-three percent had no college degree, a curriculum without any rational sequence of subjects, and an inadequate and decaying library.[69]

In addition, a variety of social, economic, and political pressures had been moving legal preparation away from an apprentice-based system

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Rudolph, *The American College and University: A History* (1962); Laurence R. Veysey, *The Emergence of the American University* (1965).

63. *See* Geiger, *supra* note 26, at 4.

64. *See* Rudolph, *supra* note 62, at 244–45.

65. *See* Edward Shils, *The Order of Learning in the United States: The Ascendancy of the University*, in Oleson & Voss, *supra* note 26, at 19, 19.

66. *See* Rudolph, *supra* note 62, at 241.

67. *See id.* at 272–86.

68. *See* Lawrence M. Friedman, *History of American Law* 608–12 (2d ed. 1985); Alfred Z. Reed, *Training for the Public Profession of Law* 273 (1921); Stevens, *supra* note 61, at 24–25.

69. James B. Ames, *Christopher Columbus Langdell*, in *Lectures in Legal History* 467, 477 (1913), *quoted in* Eric M. Holmes, *Education for Competent Lawyering—Case Method in a Functional Context*, 76 Colum. L. Rev. 536, 543 (1976).

and toward the law school[70]—particularly the university-affiliated law school.[71]

Existing law schools, however, were a marginal part of the academy,[72] and a failure to adapt to the new university ideal risked further marginalization within the burgeoning intellectual community. Indeed many, if not most, law schools retained a second class status well into the 20th century.[73]

C. *Langdell and Law As a Science*

Describing law as a science helped American law schools enter the intellectual mainstream by appealing to the dominating intellectual spirit.[74] In Langdell's words:

> I have tried to do my part towards making the teaching and study of law [at Harvard] worthy of a university; toward making [Harvard] . . . a true university, and the law school not the least of its departments . . . .
>
> To accomplish these objects, so far as they depended upon the law school, it was indispensable to establish at least two things—that law is a science, and that all the available materials of that science are contained in printed books. If law be not a science, a university will consult its own dignity in declining to teach it. If it be not a science, it is a species of handicraft, and may best be learned by serving an apprenticeship to one who practises it.[75]

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70. *See* Friedman, *supra* note 68, at 606–07; Stevens, *supra* note 61, at 20–24.

71. *See* Brubacher & Rudy, *supra* note 62, at 205; Friedman, *supra* note 68, at 606–07; James W. Hurst, *The Growth of American Law* 259–60 (1950); Reed, *supra* note 68, at 151, 189; Thomas C. Grey, *Langdell's Orthodoxy*, 45 U. Pitt. L. Rev. 1, 37–38 (1983); John H. Schlegel, *Langdell's Legacy or the Case of the Empty Envelope*, 36 Stan. L. Rev. 1517, 1521–25 (1984).

72. *See* Friedman, *supra* note 68, at 608–09; Stevens, *supra* note 61, at 35–36.

73. *See* Stevens, *supra* note 61, at 37.

74. *See* Josef Redlich, *The Common Law and the Case Method in American University Law Schools* 17 (1914); Stevens, *supra* note 61, at 51–52; William Twining, *Karl Llewellyn and the Realist Movement* 13 (1973); Holmes, *supra* note 69, at 543–44; Schlegel, *supra* note 61, at 57–58; Marcia Speziale, *Langdell's Concept of Law As Science: The Beginning of Anti-Formalism in American Legal Theory*, 5 Vt. L. Rev. 1, 25–26 (1980).

Langdell also tightened admission and graduation requirements, and hired full-time professional faculty. *See* Friedman, *supra* note 68, at 612; Stevens, *supra* note 61, at 38.

75. C.C. Langdell, Address at the 250th Anniversary of Harvard University (Nov. 5, 1886), *in* 3 L.Q. Rev. 118, 123–24 (1887).

What, however, did Langdell mean by asserting that law is a science? Consider the following three statements:

(1) Law, considered as a science, consists of certain principles or doctrines. To have such mastery of these as to be able to apply them with constant facility and certainty to the ever tangled skein of human affairs, is what constitutes a true lawyer . . . . Each of these doctrines has arrived at its present state by slow degrees . . . . [I]t is a growth, extending in many cases through centuries. This growth is to be traced in the main through a series of cases; and much the shortest and best, if not the only way of mastering the doctrine effectually is by studying the cases in which it is embodied. But the cases which are useful and necessary for this purpose at the present day bear an exceedingly small proportion to all that have been reported. The vast majority are useless, and worse than useless, for any purpose of systematic study. Moreover the number of fundamental legal doctrines is much less than is commonly supposed; the many different guises in which the same doctrine is constantly making its appearance, and the great extent to which legal treatises are a repetition of each other, being the cause of much misapprehension. If these doctrines could be so classified and arranged that each should be found in its proper place, and nowhere else, they would cease to be formidable from their number. It seemed to me, therefore, to be possible to take such a branch of the law as Contracts, for example, and without exceeding comparatively moderate limits, to select, classify, and arrange all the cases which had contributed in any important degree to the growth, development, or establishment of any of its essential doctrines; and that such work could not fail to be of material service to all who desire to study that branch of law systematically and in its original sources.[76]

(2) [L]aw is a science and . . . all the available materials of that science are contained in printed books. . . .

. . . [The library] is to us all that the laboratories of the university are to the chemists and physicists, the museum of natural history to the zoologists, the botanical garden to the botanists.[77]

76. C.C. Langdell, *A Selection of Cases on the Law of Contracts* at viii-ix (2d ed. 1879).

77. Langdell, *supra* note 75, at 124.

> (3) The opinion has . . . been prevalent that [law] is incapable of being taught as a science; and, though the correctness of this opinion will not be admitted by those who represent this School; it may be supported by plausible arguments. *Law has not the demonstrative certainty of mathematics;* nor does one's knowledge of it admit of . . . simple and easy tests, as in case of a dead or foreign language; *nor does it acknowledge truth as its ultimate test and standard, like natural science*; nor is our law embodied in a written text, which is to be studied and expounded, as is the case with the Roman law and with some foreign systems.[78]

Commentators interpret such quotations in various ways. Some find Langdell's assertion to be that law as a science is deductive.[79] Others find Langdell's assertion to be that law as a science is inductive.[80] Still others find something of a mixture.[81] Finally, some commentators see Langdell as an opportunist willing to adopt a certain mode of expression when convenient for the implementation of his overriding goal: the development of comprehensive, university-based legal education.[82]

A premise of this Article is that the core of understanding (i.e. science) is classification. There is no attempt here to provide any classification scheme for law. This Article merely asserts that

78. C.C. Langdell, *Annual Report on the Law School*, in *Fifty-Second Annual Report of the President of Harvard College 1876–77*, at 96–97 (1878), *quoted in* Anthony Chase, *Origins of Modern Professional Education: The Harvard Case Method Conceived As Clinical Instruction in Law*, 5 Nova L.J. 323, 358 (1981).

79. *See* Friedman, *supra* note 68, at 617; Banner, *supra* note 45, at 253 n.33; Dow, *supra* note 4, at 708; *cf.* Paul D. Carrington, *The Missionary Diocese of Chicago*, 44 J. Legal Educ. 467, 468 (1994); Willard Hurst, *Changing Responsibilities of the Law School: 1869–1968*, 1968 Wis. L. Rev. 336, 336–37; Catherine P. Wells, *Holmes on Legal Method: The Predictive Theory of Law As an Instance of Scientific Method*, 18 S. Ill. U. L.J. 329, 330 (1994).

80. *See* Redlich, *supra* note 74, at 15–16; Arthur E. Sutherland, *The Law at Harvard: A History of Men and Ideas, 1817–1967*, at 174–75 (1967); Holmes, *supra* note 69, at 546; Gary Minda, *One Hundred Years of Modern Legal Thought from Langdell and Holmes to Posner and Schlag*, 28 Ind. L. Rev. 353, 358–61 (1995); Speziale, *supra* note 74, at 29.

81. *See* Stevens, *supra* note 61, at 52–53; Grey, *supra* note 71, at 16; M.H. Hoeflich, *Law & Geometry: Legal Science from Leibniz to Langdell*, 30 Am. J. Legal Hist. 95, 119-20 (1986); *cf.* Nancy Levit, *Listening to Tribal Legends: An Essay on Law and Scientific Method*, 58 Fordham L. Rev. 263, 275–76 (1989); John Veilleux, Note, *The Scientific Model in Law*, 75 Geo. L.J. 1967, 1974–76 (1987).

Some authors are not entirely clear in their descriptions of Langdell. For an example, see Gilmore, *supra* note 59, at 42–43.

82. *See* Chase, *supra* note 78, at 342, 358–59.

classification is the heart of the scientific component of law as a discipline.[83]

The idea that law involves classification is nothing new. Classification is an important part of both the common and civil law traditions.[84] Even Oliver Wendell Holmes, one of Langdell's harshest critics,[85] was eager to provide a rational scheme of classification.[86] Moreover, classification is an important part of the jurisprudential movements that have shaped current American legal academics.[87]

Classification is at the heart of Langdell's notions of law as a science as described in the first two numbered quotations above.[88] In this sense, Langdell's efforts are commendable, and Gilmore is wrong to characterize Langdell's ideas as "absurd" or "mischievous" or "rooted in

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83. For an exhaustive and provocative account of classification in law, see Jay M. Feinman, *The Jurisprudence of Classification*, 41 Stan. L. Rev. 661 (1989).

84. For an overview of various schemes of legal classification, see 5 Roscoe Pound, *Jurisprudence* 5–75 (1959).

85. Holmes called Langdell a "legal theologian." Book Review, 14 Am. L. Rev. 233, 234 (1880) (reviewing Langdell, *supra* note 76). This review is unsigned, but Holmes is understood to be the author. *See* Mark D. Howe, *Introduction* to Oliver Wendell Holmes, *The Common Law* at xxii n.9 (Mark D. Howe ed., 1963) (1881). Some of Langdell's students saw him as anything but a theologian. *See* Franklin G. Fessenden, *The Rebirth of the Harvard Law School*, 33 Harv. L. Rev. 493 (1920).

86. *See* Howe, *supra* note 85, at xiv; Wells, *supra* note 79, at 332.

87. This is most obviously true in work predating appeals to the social sciences. *See* James E. Herget, *American Jurisprudence, 1870-1970: A History* 1–116 (1990).

According to G. Edward White, the social science-based movements include: (1) the sociological jurisprudence movement, (2) the legal realist movement, (3) the law, science, and policy movement, (4) the legal process movement, and (5) the law and society movement. *See* G. Edward White, *From Realism to Critical Legal Studies: A Truncated Intellectual History*, 40 Sw. L.J. 819 (1986); G. Edward White, *The Evolution of Reasoned Elaboration: Jurisprudential Criticism and Social Change*, 59 Va. L. Rev. 279 (1973); G. Edward White, *From Sociological Jurisprudence to Realism: Jurisprudence and Social Change in Early Twentieth Century America*, 58 Va. L. Rev. 999 (1972).

These later movements embraced classification as well. *See* 5 Pound, *supra* note 84, at 21 (considering sociological jurisprudence movement); Wilfred E. Rumble, Jr., *American Legal Realism: Skepticism, Reform, and the Judicial Process* 31 (1968) (considering legal realist movement); Twining, *supra* note 74, at 26–40 (considering legal realist movement); Harold D. Lasswell & Myres S. McDougal, *Legal Education and Public Policy: Professional Training in the Public Interest*, 52 Yale L.J. 203, 243–89 (1943) (considering law, science, and policy movement); Robert B. Yegge, *President's Message: Law and Sociology*, 4 L. & Soc'y Rev. 327–28 (1970) (considering law and society movement); Henry M. Hart, Jr. & Albert M. Sacks, *The Legal Process* 141–60 (tentative ed. 1958) (considering legal process movement).

88. *See* Twining, *supra* note 74, at 12; Wells, *supra* note 79, at 332; Gary J. Aichele, *Legal Realism and Twentieth Century American Jurisprudence: The Changing Consensus* 24 (1983) (unpublished Ph.D. dissertation, University of Virginia).

European thought may have influenced Langdell's thinking. *See* John H. Merryman, *The Civil Law Tradition* 62, 66–67 (2d ed. 1985).

error."[89] Indeed, the stance taken in the third quotation, which specifically distinguishes law from mathematics and the natural sciences, may represent a realization that the nature of legal classification is subtle, complex, and unique to law.

On the other hand, Langdell's views illustrate one of the potential pitfalls in doing interdisciplinary work. In particular, to the extent that his comments represent not merely an attempt to bring law within the intellectual mainstream but a genuine adoption of what would be described colloquially as the "scientific model,"[90] they indicate a skewed attitude that is typical of much recent intellectual history.[91]

IV. LAW AND THE FOUNDATIONAL CRISIS IN MATHEMATICS: A CASE STUDY

> *Traditional epistemology, with its belief in the existence of transcendent, objective truth, has been replaced in the twentieth century by a "new epistemology," which rejects a belief in objective truth and the claims of certainty that traditionally follow. The new epistemology describes a broad shift in the theory of knowledge; it has permeated such . . . fields as mathematics.*
>
> *Joan Williams*[92]

> *Platonism dies very hard—and nowhere harder than among mathematicians. . . . Perhaps mathematics is actually in this sense the least "modern" of modern endeavors.*
>
> *Harry Grant*[93]

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89. *See supra* text accompanying note 59.

90. For an overview of possible such approaches to law, see Funk, *supra* note 16.

91. For a discussion of the geometry-based models of disciplines that were prevalent throughout much of the 17th, 18th, and early 19th centuries, see Morris Kline, *Mathematics in Western Culture* 322–39 (1953). For a discussion of the natural science-based models that are typical of the period since the early 19th century, see Tom Sorell, *Scientism: Philosophy and the Infatuation with Science* 1–23 (1991). This shift in models tracks the larger intellectual shift from rationalism toward empiricism. *See* Edward M. Burns, *Western Civilizations: Their History and Their Culture* 445–61, 637–49 (3d ed. 1949).

Langdell also has been criticized for focusing on law as a science to the exclusion of law as a technology. *See* Carrington, *supra* note 79.

92. Williams, *supra* note 12, at 430–31.

93. Harry Grant, *What Is Modern About "Modern" Mathematics?*, Mathematical Intelligencer, Summer 1995, at 62, 65.

A. *Introduction*

The problems with invocations of the current mathematical crisis stem from a lack of attention to the intellectual history of the crisis. Accordingly, section B provides this history. Using this history, section B then presents a detailed discussion of the current crisis and the relevant mathematics. Using this discussion, section B closes with an analysis of the first of the three types of problems mentioned in the Introduction to this Article—the accuracy of descriptions appearing in the legal literature. Section B accomplishes one other task by emphasizing one aspect of the interaction of the disciplinary and mathematical pieces of this Article. In particular, it illustrates what is involved in overcoming the first difficulty in doing meaningful legal interdisciplinary research: gaining a sufficient understanding of what is often a foreign discipline. Section C also accomplishes two tasks. First, it considers the other two problems mentioned in the Introduction: the facile application of the mathematics to law, and the use of the current crisis as support for a general intellectual skepticism. Second, it develops the other aspects of the interaction of the mathematical and disciplinary pieces of this Article. In particular, section C uses the disciplinary framework to see how the crisis initially made its way into the legal literature, and it illustrates the second difficulty in doing meaningful interdisciplinary legal research: employing another discipline in a manner that reflects both its relevance to, and separateness from, law.

B. *The Current Foundational Crisis in Mathematics*

1. *Preliminary Comments*

Section B is the most difficult segment of the Article as it contains almost all of the technical material. This subsection presents a synopsis. It will not hurt to have the basic story repeated more than once. To further facilitate understanding, the remaining subsections of section B have been organized so that the text contains a discussion for the educated lay reader, while the footnotes contain numerous references and some technical details.

The mathematics is important for developing the disciplinary points of the Article. Section B illustrates what is entailed in overcoming the first hurdle in doing meaningful interdisciplinary work—namely, gaining an understanding of what is often a foreign discipline. Such an understanding also helps address the second hurdle because a detailed study of a part of another discipline often reveals that discipline's relevance and separateness.

After studying section B, the reader should understand that the current foundational crisis in mathematics is just the most recent in a sequence of three such crises.[94] These crises have framed the evolution of mathematics for some 2500 years.[95] The sequence of mathematical crises can be described in a variety of ways. Section B provides a narration that both reflects traditional discussions and is well suited for the purposes of examining the foundational crises and their treatment by legal scholars. More specifically, the crises are discussed in terms of three basic issues: the (apparent) certainty of mathematics as a science; the (apparent) soundness of mathematics as a technology; and the mystery of the infinite.[96] These issues are traditional starting points for the philosophy of mathematics,[97] and the first two have made mathematics a central part of the Western intellectual tradition.[98] In telling the story, certain shortcuts are unavoidable given that events span 2500 years.

The reader also should understand that these mathematical crises are not isolated events but parts of much larger intellectual currents. The first crisis was part of the maturation of Greek culture that took place in the sixth, fifth, and fourth centuries before the common era. The second crisis was part of the appearance of the Enlightenment, and the third crisis is part of the comprehensive reevaluation of the Western intellectual tradition described at the end of part II.

Moreover, the reader should understand that the evolution of mathematics was shaped, *not* retarded, by these crises. Subsection 2 describes the first crisis. The Greeks were the first to undertake a

94. At this point, some readers may begin to wonder whether Thomas Kuhn's concept of paradigm shift is applicable to mathematics. The debate rages! For a number of articles on the issue, see *Revolutions in Mathematics* (Donald Gillies ed., 1992).

95. Some commentators assert that the use of the word "crisis" is misleading, if not inaccurate. *See* Salomon Bochner, *The Role of Mathematics in the Rise of Science* 138–42 (1966).

96. Some readers may wonder why the third question is not, "What is the nature of mathematics as an art?" As explained immediately below, the choice here is to reflect traditional discussions, and traditional presentations do not describe the crises in artistic terms. From time to time, however, the rudiments of such a treatment are indicated. *See infra* notes 113, 200; text accompanying notes 182–83.

97. *See* Stephan Körner, *The Philosophy of Mathematics* 9–12 (1960). The philosophy of mathematics *per se* is discussed no more than is necessary for the purposes of this Article.

98. *See* Mary Tiles, *Mathematics and the Image of Reason* 1–6 (1991); Judith V. Grabiner, *The Centrality of Mathematics in the History of Western Thought*, 61 Mathematics Mag. 221 (1988); Philip Kitcher & William Aspray, *An Opinionated Introduction*, in *History and Philosophy of Modern Mathematics* 3, 17 (William Aspray & Philip Kitcher eds., 1988).

sustained investigation of mathematics, yet these very investigations created a number of perplexing questions concerning the three basic issues listed above. In trying to grapple with these questions, the Greeks were led to a number of innovations, including the introduction of axiomatics, the development of various techniques for dealing with the infinite, and the formulation of some fundamental philosophical positions. Subsection 3 describes the second crisis. During the post-Renaissance development of the calculus, mathematicians encountered many of the same questions faced by the Greeks. Once again the result was a number of important developments, including a more robust attitude towards the infinite and the further elaboration of philosophical positions. Subsection 4 introduces the current crisis in which the same questions appeared yet again in the context of non-Euclidean geometry and set theory. For the purposes of this Article, the most important development to emerge from this crisis was the attempt to deal with the basic issues in terms of well-delineated philosophies of mathematics. Although such philosophies had the advantage of focusing discussion in a way not theretofore possible, their diversity led to an internal turmoil that actually threatened to tear the discipline apart. David Hilbert stepped into this maelstrom and attempted to unite the then-contending philosophic schools through the so-called Hilbert Program. This Program involves a careful balancing of the various positions. Unfortunately, Gödel's Theorems deal it a serious, if not fatal, blow. In essence, they turn Hilbert's balancing against itself. The reader must understand that this balancing is at the core of Gödel's Theorems. Indeed, the theorems as stated are of limited scope and application. Moreover, attention to their intellectual history is crucial for a full understanding of their content, context, and relevance.

The stories of non-Euclidean geometry, Hilbert, and Gödel comprise the most difficult mathematical topics, and they are segregated into subsection 5. That subsection also contains the analysis of the accuracy of the invocations of the current crisis appearing in the legal literature. Subsection 6 has some final comments.

2. *The First Crisis*

The story begins with the Greeks. Earlier cultures had considered the practical (i.e. technological) component of mathematics, but the consideration of mathematics as a science was largely a Greek development.[99] Carl Boyer describes it as follows:

> [T]here is an obvious change in spirit in both science and mathematics, as these developed in Greece. The human mind was "discovered" as something different from the surrounding body of nature and capable of discerning similarities in a multiplicity of events, of abstracting these from their settings, generalizing them, and deducing therefrom other relationships consistent with further experience.[100]

Such a capability, however, creates the possibility of what Raymond Wilder has called the "curious duality" of the world of mathematics and the world of the senses.[101] This duality raises questions about the relationship of mathematics as a science to mathematics as a technology. Why, for example, would preeminently rational activities such as abstraction and deduction yield information about the world of immediate sense impressions?[102]

The Pythagoreans attempted to skirt this duality through an enigmatic atomistic philosophy asserting that whole numbers (1, 2, 3, . . .) make up the essence of being. Such a position might sound bizarre today, but it was not so strange in an era in which philosophical speculations focused on the nature, rather than the likeness of things, and in which mathematics was perceived to be the basis of areas as diverse as trade and music.[103] Two mathematical events, however, raised doubts about such an approach.

First, the Pythagoreans themselves made the unsettling discovery that whole numbers are inadequate to compare the diagonal of a unit square with its sides. In modern terms, there is no rational number whose square is two.[104] Indeed, the traditional story is that the Pythagoreans pledged not to divulge this discovery, and the person who broke his word was murdered for his indiscretion.[105]

99. Carl B. Boyer, *The History of the Calculus and Its Conceptual Development* 14–16 (1949).

100. *Id.* at 16.

101. *See* Raymond L. Wilder, *Evolution of Mathematical Concepts* 152–53 (1968).

102. *See* Margaret E. Baron, *The Origins of the Infinitesimal Calculus* 18 (1969).

103. For a discussion of the complex Pythagorean position, see Edward Maziarz & Thomas A. Greenwood, *Greek Mathematical Philosophy* 10–23 (1968).

104. *See* Carl B. Boyer, *A History of Mathematics* 79 (1968). There is some doubt whether the argument provided *supra* note 31 was the argument employed by the Pythagoreans themselves. For a suggestion about the original argument, see Boyer, *supra*, at 80–81.

105. *See* Eli Maor, *To Infinity and Beyond: A Cultural History of the Infinite* 46 (1987).

At roughly the same time, the famous Zeno Paradoxes indicated other difficulties with mathematical atomism. As a fundamental matter, the Paradoxes were part of an overarching debate among certain pre-Socratic philosophers about the fundamental nature of reality.[106] In particular, the Paradoxes supported the Parmenidian tenet of permanence by attacking the opposing Heraclitian doctrine of change. The most well-known of the Paradoxes were specifically aimed at exposing the illusory nature of motion through *reductios* based on assumptions about the finite or infinite divisibility of space and time.[107] In trying to separate the sensible change from the rational permanence, however, these motion-based Paradoxes also represented a general attack on the Pythagorean metaphysics.[108]

Both of these events raised serious issues about the infinite.[109] The discovery of the irrationals forced an awareness of the limitations of a purely finite approach to describing magnitudes, and the Paradoxes indicated the subtle difficulties that could result by too quickly introducing notions of the infinite.[110]

The resulting intellectual shock is called the first foundational crisis in mathematics.[111] Three developments emerged from this crisis.

First, the Greeks introduced the axiomatic method to mathematics.[112] They hoped that axiomatization would provide a secure foundation and

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106. For a brief introduction to the pre-Socratic debates, see Lavine, *supra* note 26, at 24–25.

107. For the purposes of this Article, perhaps the best discussion of this point is contained in Kline, *Mathematical*, *supra* note 29, at 34–37. At that time, there were two theories of motion. One, based on the notion that time and space were infinitely divisible, described motion as continuous and smooth. The other, based on the notion that time and space were made up of indivisible units, described motion as a collection of small jerks. Zeno's Paradoxes attacked each of these theories through a *reductio*. The "Dichotomy" was one of the Paradoxes aimed at the first theory: to travel from *A* to *B* one had to reach the midpoint *M* between *A* and *B*, but to reach *M* one had to reach the midpoint *M'* between *A* and *M*, and so forth, so that the very beginning of motion was impossible. The "Arrow" was one of the Paradoxes aimed at the second theory: an arrow in flight is really at a standstill because at each of the indivisible instants of time it occupies a definite position in space.

Zeno's extensive use of the *reductio* technique led Aristotle to credit him as the creator of the dialectic method. *See* 1 Thomas Heath, *A History of Greek Mathematics* 273 (1921).

108. *See* Maziarz & Greenwood, *supra* note 103, at 63–64. *But see* 1 Heath, *supra* note 107, at 271–83 (asserting that Paradoxes had nothing to do with Pythagorean mathematical metaphysics).

109. *See* 2 Kramer, *supra* note 29, at 298.

110. *Id.* There is some controversy about the exact mathematical significance of the paradoxes. *See* Baron, *supra* note 102, at 22–25 (describing assertion by some that paradoxes had no mathematical significance).

111. *See* Fraenkel et al., *supra* note 3, at 13; Wilder, *supra* note 101, at 109.

112. *See* Wilder, *supra* note 101, at 97–98. For a description of the Greek axiomatic approach, see *infra* text accompanying note 186.

make it easier to produce new results.[113] Algebra/arithmetic was described largely in geometric terms,[114] and geometry was constituted as a collection of propositions derived from specified assumptions and definitions. The resulting system included techniques that apparently avoided the most immediately troublesome of the mathematical difficulties associated with irrationals and the paradoxes.[115]

Second, Greek mathematicians developed a bifurcated procedure for working with the infinite. Results had to be established with geometric techniques that used only so-called potential infinity, but the Greeks were willing to use so-called actual infinity as an investigatory heuristic.[116] One mathematical commentator describes the difference between potential and actual infinity as follows:

> The former involves a process that can be repeated again and again without end, but which, at any given stage, still encompasses only a finite number of repetitions. . . . The actual infinite, on the other hand, involves a process which has already acquired . . . an infinite number of repetitions.[117]

Another describes it as follows:

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113. *See* Stephen F. Barker, *Philosophy of Mathematics* 24–25 (1964); Richard J. Trudeau, *The Non-Euclidean Revolution* 2–4 (1987). Some commentators see aesthetics at work here as well. *See* Barker, *supra*, at 25.

114. *See* Boyer, *supra* note 104, at 84–85.

115. *See id.* at 98–102.

116. *See* Baron, *supra* note 102, at 46. For a discussion of how Zeno's paradoxes led to the exaltation of potential over actual infinity, see Mary Tiles, *The Philosophy of Set Theory: An Introduction to Cantor's Paradise* 12–21 (1989).

117. Maor, *supra* note 105, at 54–55.

For a rough understanding of the bifurcation referred to in the text, consider the problem of finding the area A of a plane region R. The typical application of the actual infinite heuristic begins with an appropriately chosen region R' of known area A'. The regions R and R' are each divided into an infinite number of pieces. The R pieces are put in a one-to-one correspondence with the R' pieces in such a way that when R and R' are compared piece by piece it is evident that the total areas of R and R' differ by a multiplicative factor of c. That is, it is evident that $A = cA'$. Since this comparison process consists of comparing all of the pieces, it involves the actual infinite. Once the number c is thus obtained, the usual potential infinite approach consists of showing that the inequalities $A < cA'$ and $A > cA'$ are impossible. For example, suppose that $A < cA'$, say $cA' - A = e$. A contradiction would be obtained by inscribing inside R a finite number of non-overlapping regions of known characteristics whose total area is larger than $cA' - e$. This is a contradiction because there would be inscribed in R a collection of non-overlapping regions whose total area is greater than $cA' - e = A$, the area of R. The actual number of regions would depend, *inter alia*, on the purported difference $e = cA' - A$, but would in every case be finite. Thus, this process involves only the potential infinite. A similar argument would be used to show that $A > cA'$ is impossible. For a specific example, see Kline, *Mathematical*, *supra* note 29, at 110–14. *See also* Baron, *supra* note 102, at 34–50.

> [The Greeks] were prepared to accept that, given any number, however large, there would always be a larger number, and that given any line, however long, it could always be extended further. They were not prepared to accept the concept of an infinite collection of numbers nor that of a line of infinite magnitude, that is, whilst the concept of something being "potentially" infinite was acceptable to them, they carefully avoided . . . objects that were "actually" infinite.[118]

The hesitant attitude towards the infinite and the lack of a mature and separate algebra/arithmetic had a number of specific consequences. The development of analysis/calculus, for example, was postponed for 2000 years.[119] In addition, mathematics started down the path to non-Euclidean geometry. This part of the story is described in more detail in subsection 5.a. below. For now, the reader should understand that the starting point was parallelism. On the one hand, assertions about parallelism would be suspect to the extent that they did not involve finite figures or finite parts of figures; that is, such statements were problematic to the extent that they conceived of a straight line as an infinite whole. On the other hand, many results seemed to require some assumptions about parallelism. Euclid's solution was to develop a framework that arguably dealt only with potentially infinite figures. The cornerstone was his famous Fifth Postulate. Even this framework caused immediate concern, and mathematicians attempted to rework it. Efforts focused on replacing the Fifth Postulate with something less objectionable or to derive it from the remaining assumptions. None of these efforts were successful. Instead, mathematicians were led to non-Euclidean geometry.

The third development to emerge from the first crisis was an embracing of the mathematical dualism described above.[120] Dualism, however, raises two concrete issues: the certainty of mathematics as a science and the soundness of mathematics as a technology. Greek thought offered a variety of perspectives on these issues. For Plato,[121] mathematics inhabited a world independent of perception, yet having a

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118. *See* Graham Flegg, *Numbers: Their History and Meaning* 256 (1983).

119. *See* Boyer, *supra* note 99, at 59–60; Maor, *supra* note 105, at 3.

This episode suggests a synthesis of the objective and subjective in mathematics—namely, one can accept the subjective forces that shaped the boundaries of Greek mathematics while simultaneously appreciating the objectivity of its contents. For a general discussion of the forces that shape mathematical evolution, see Wilder, *supra* note 101.

120. *See* Wilder, *supra* note 101, at 152–53.

121. For an introduction to Plato's mathematical views, see Körner, *supra* note 97, at 14–18.

real and eternal existence. The certainty of mathematics as a science rested on truth apprehended by carefully enunciated reason. The soundness of mathematics as a technology rested on a certain type of approximation of the mathematical by the sensible.[122] Mathematical objects had no such existence for Aristotle;[123] they represented mental idealizations of the sensible world. As with Plato, the soundness of mathematics as a technology rested on some kind of approximation, but the certainty of mathematics rested on a rigorous notion of logical necessity.[124] Euclid's *Elements* could be read with either a Platonic or Aristotelian gloss.[125]

This mathematical crisis did not exist in a vacuum. Western thought itself was undergoing a profound reformulation with the maturation of Greek culture that took place in the sixth, fifth, and fourth centuries before the common era.[126] The emergence of, and reaction to, the first foundational crisis was an integral part of this larger intellectual current.[127]

3. *The Second Crisis*

By the early 17th century, the mathematical obstacles to the development of analysis/calculus had been removed. A more mature and separate algebra/arithmetic appeared,[128] and the amalgamation of algebra and geometry into analytic geometry allowed mathematicians to attack a problem with both the symbolic, rote calculation approach of algebra and

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122. There is some dispute as to whether the world of mathematics was part of Plato's world of forms, or intermediate to it and the world of immediate sense impressions. *Compare id.* at 15 *with* Maziarz & Greenwood, *supra* note 103, at 135.

123. For a brief introduction to Aristotle's views, see Körner, *supra* note 97, at 18–21.

124. There is some question about how much the mathematical Aristotle in fact differed from the mathematical Plato. *See id.* at 18–19; Francois Lasserre, *The Birth of Mathematics in the Age of Plato* 32 (1966).

125. *See* Maziarz & Greenwood, *supra* note 103, at 229–30. There are many editions of Euclid. One of the more popular is by Thomas Heath.

126. *See* Burns, *supra* note 102, at 103, 121–31.

127. For an overview of the intellectual context of Greek mathematics, see Maziarz & Greenwood, *supra* note 103. *See also* Israel Kleiner, *Rigor and Proof in Mathematics*, 64 Mathematics Mag. 291, 293 (1991) (describing emergence of axiomatic method in larger context).

128. This development was due in large part to Arab and Indian mathematicians. For an overview of their contributions, see Boyer, *supra* note 104, at 229–69.

This algebra/arithmetic did not, however, take the axiomatic form of geometry. It was presented as a collection of techniques of calculation. Axiomatization came much later. *See* Barker, *supra* note 113, at 56–57.

the visual, intuitive approach of geometry.[129] Moreover, mathematicians began to shed the hesitancy that characterized the Greek attitude towards infinitary techniques.[130] There is little wonder that the last half of the 17th century saw the development of the calculus.[131]

In the early 18th century, however, it became clear that there were considerable problems with the calculus as a science despite its astounding success as a technology. Greek rigor, as exemplified by the axiomatic method, had not been carefully pursued.[132] A number of conceptual questions arose that, as Boyer notes, were "in the last analysis equivalent to those that Zeno had raised well over two thousand years previously and were based on questions of infinity and continuity."[133] The reappearance of these neo-Zenonic questions should not be surprising given that much of the early calculus concerned the study of motion.[134] Many mathematicians used technological success to deal with

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129. For a brief look at analytic geometry, see Kline, *Mathematical*, *supra* note 29, at 302–24. The key, of course, is the use of a coordinate system to represent geometric points in terms of numbers. In this way, geometric figures can be expressed in terms of numerical conditions satisfied by the coordinates of their constituent points. Such a representation, for example, yields the equations for lines, parabolas, ellipses, and circles.

130. *See* Boyer, *supra* note 99, at 186. For discussions of the infinitary techniques in the century preceding the work of Isaac Newton and Gottfried Leibniz, see Baron, *supra* note 102, at 90–149; Boyer, *supra* note 99, at 96–186.

131. *See* Boyer, *supra* note 99, at 187–88. For discussions of the evolution of the calculus, see Boyer's work as well as Baron, *supra* note 102; Charles H. Edwards, Jr., *The Historical Development of the Calculus* (1979).

132. *See* Kline, *Mathematical*, *supra* note 29, at 383–89.

133. Boyer, *supra* note 99, at 267. *See also* Tobias Dantzig, *Number: The Language of Science* 133–34 (1954); Dirk J. Struik, *A Concise History of Mathematics* 150 (3d rev. ed. 1967).

134. Suppose that a particle moves along in a straight line on which a coordinate system already has been introduced. The instantaneous velocity of a particle is identified with the derivative of the function representing its position in terms of time. Consider then George Berkeley's assertion that Newton's approach to differentiation was flatly inconsistent. Suppose Newton wished to find the velocity of a particle whose coordinate at time x is given by the function f where $f(x) = x*x$. In essence, Newton's technique consisted of finding the derivative of f at x in two steps. First, Newton considered $(f(x+h)-f(x))/h$, the average velocity over the time interval from x to $x+h$. He simplified this expression by expanding $(x+h)*(x+h)$ to $x*x + 2*x*h + h*h$, subtracting $x*x$, and dividing by h to obtain the expression $2*x + h$ for the average velocity. In the second step, Newton treated h as an "evanescent quantity" to obtain the value $2*x$ for the instantaneous velocity at time x. Berkeley pointed out that the first step assumes that h is a non-zero quantity while the second step seems to assume that it is zero. *See* Boyer, *supra* note 99, at 225–27. In Berkeley's criticism, one sees the ghosts of Zeno's old problems with the nature of the divisibility of time. Given Berkeley's critique, the reader may wonder how mathematicians were able to avoid widespread inconsistencies in their results. For a discussion of this point, see Judith V. Grabiner, *The Origins of Cauchy's Rigorous Calculus* 22 (1981).

or even ignore these questions,[135] but there was increasing concern about the foundations of the calculus.[136] Indeed, in 1784 the Berlin Academy proposed the question of the foundations of the calculus as one of its celebrated mathematical prize problems.[137]

The resulting intellectual shock has been called the second foundational crisis.[138] Three developments emerged from this crisis.

First, mathematicians began to pay more attention to mathematical rigor. In particular, the early 19th century development of the limit concept dealt with the most immediately troublesome of the difficulties associated with the neo-Zenonic criticisms.[139] In this sense, the introduction of limits was analogous to the techniques the Greeks introduced to overcome the most immediately troublesome of the mathematical difficulties associated with irrationals and the paradoxes.[140]

Second, some mathematicians relaxed the strict bifurcated approach to infinity. Nineteenth century mathematicians viewed the limit concept as involving potential as opposed to actual infinity.[141] Nonetheless, a number of mathematicians openly had embraced the use of actual infinity before the limit concept was formulated.[142]

For general discussions of problems with the foundations of the calculus, see Boyer, *supra* note 99, at 224–67; H.J.M. Bos, *Newton, Leibniz, and the Leibnizian Tradition*, in *From the Calculus to Set Theory 1630–1910: An Introductory History* 49, 86–89 (I. Grattan-Guinness ed., 1980).

135. *See* Grabiner, *supra* note 134, at 16–17; Kline, *Mathematical*, *supra* note 29, at 618.

136. *See* Kline, *Mathematical*, *supra* note 29, at 947. For a discussion of early non-technological approaches, see Grabiner, *supra* note 134, at 31–37.

137. See Grabiner, *supra* note 134, at 40–43.

138. *See* Fraenkel et al., *supra* note 3, at 13; *see also* Howard Eves & Carroll V. Newsom, *An Introduction to the Foundations and Fundamental Concepts of Mathematics* 296 (rev. ed. 1965); Wilder, *supra* note 101, at 109.

139. With respect to Berkeley's criticism of differentiation, the limit concept made it clear that the second step in the differentiation described *supra* note 134 did not treat *h* as zero. For a thorough discussion of the development of the limit concept, see Grabiner, *supra* note 134.

140. *See* Struik, *supra* note 133, at 149. Indeed, the limit concept itself can be traced to some of these earlier techniques. *See* Boyer, *supra* note 99, at 271.

141. *See* Boyer, *supra* note 99, at 267, 274–75 (describing views of Augustin-Louis Cauchy); Ettore Carruccio, *Mathematics and Logic in History and Contemporary Thought* 238 (1964) (same); Joseph W. Dauben, *Georg Cantor: His Mathematics and Philosophy of the Infinite* 96–97 (1979) (describing views of Cantor); Maor, *supra* note 105, at 55 (describing views of Carl Friedrich Gauss).

Philosopher Alexander George has asserted that from a modern perspective, the limit concept cannot be said to choose between potential and actual infinity. Telephone Interview with Alexander George, Professor of Philosophy, Amherst College (1994).

142. *See* Boyer, *supra* note 99, at 239–42. Such an attitude was presaged by certain late medieval philosophers. *See* Boyer, *supra* note 104, at 292–93.

Third, dualist stances were further elaborated. In particular, Gottfried Leibniz and Immanuel Kant advanced important new positions. Leibniz disagreed with both Platonic and Aristotelian views and held that mathematics as a science had nothing to do with eternal objects, idealized objects, or objects of any kind.[143] The certainty of mathematics was due to the tautological nature of mathematical propositions themselves. This, however, opened a huge gap between mathematics as a science and mathematics as a technology. Leibniz filled this gap with a problematic theological approach.[144] For Kant, on the other hand, the certainty of mathematics as a science did not involve tautologies but the structure of space and time as revealed in terms of an *a priori* intuition.[145] Kant's view would explain the soundness of mathematics as a technology insofar as it deals with this structure. Both Leibniz and Kant reflected the bolder mathematical attitudes towards infinity. Aristotle had asserted that instances of actual infinity did not exist in the world of sense impressions; indeed, it was logically impossible that they would exist.[146] Kant agreed with the first assertion but not the second. For Kant, actual infinity was a so-called Idea of Reason—an internally consistent concept not applicable to sense experience.[147] The views of Leibniz were not always consistent, but at times he seemed to go even further than Kant, making a distinction between actual infinity, which could indeed be said to exist in nature, and the ability to conceptualize and work with this infinity, which belonged only to God.[148] Then existing mathematics could be viewed from Platonic, Aristotelian, Leibnizian, or Kantian positions.

Once again, this crisis did not exist in a vacuum. Western thought was undergoing a profound reformulation with the appearance of the Enlightenment.[149] The development of the calculus and the emergence of,

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143. For general discussions of the views of Leibniz, see 4 Copleston, *supra* note 45, at 273–94; Körner, *supra* note 97, at 21–25.

144. Stephan Körner gives an indication of Leibniz's position as follows:

> According to [Leibniz] "1 + 1 = 2" (as a statement of pure mathematics) is true on the basis of the law of contradiction, and thus in all possible worlds; whereas "1 apple and 1 apple make 2 apples" (as a statement of physics) is true in this world which God was bound to create . . . if it was to be the best of all possible worlds.

Körner, *supra* note 97, at 24.

145. For general discussions of the views of Kant, see 6 Copleston, *supra* note 45, at 235–76; Körner, *supra* note 97, at 25–31.

146. *See* Körner, *supra* note 97, at 30.

147. *See id.* at 29–31.

148. *See* Dauben, *supra* note 141, at 123–24.

149. *See* Burns, *supra* note 91, at 445–46.

and reaction to, the second foundational crisis was an integral part of this larger intellectual context.[150]

4. *The Current Crisis*

Euclid's Fifth Postulate had remained a problem.[151] For 2000 years, mathematicians had attempted either to replace it with something less objectionable or to derive it from the remaining assumptions. All such efforts had ended in failure. In the 18th century, a *reductio* approach was tried that in essence consisted of replacing the Fifth Postulate with its negation in hopes of deriving a contradiction. This approach did not work, but mathematicians were led to a number of counter-intuitive results that were interpreted as some sort of confirmation of the Fifth Postulate. It remained for the mathematicians of the first half of the 19th century to make the leap to the conclusion that these results in fact indicated the existence of non-Euclidean geometry.

Yet the jump to non-Euclidean geometry raised serious questions about the certainty of mathematics as a science and the soundness of mathematics as a technology. With respect to certainty, mathematicians were able to show that non-Euclidean geometry is no less consistent than Euclidean geometry; that is, if Euclidean geometry has no contradictions, then neither does non-Euclidean geometry. But in what scientific sense can these geometries stand side by side? And what do they say about the soundness of mathematics as a technology?

Meanwhile, work in analysis led to still more problems. The limit concept answered most of the immediate concerns that had been raised by criticisms of the calculus, but this was not the end of the story. Mathematicians had chosen to rest the limit concept on an arithmetic as opposed to a geometric basis, perhaps due in part to the uncertainties created by non-Euclidean geometry.[152] It became clear that this basis required a careful elaboration of the real number system.[153] Every approach developed in the latter part of the 19th century, however,

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150. *See id.* at 445–56; Grabiner, *supra* note 134, at 26; Kline, *supra* note 91, at 234–86.

151. The discussion in this and the preceding paragraph is developed more fully *infra* part IV.B.5.a.

152. *See* Kline, *Mathematical*, *supra* note 29, at 947–49. For other reasons, see *id.*; Tiles, *supra* note 116, at 68–84.This arithmetic basis was part of the larger so-called "arithmetization of analysis." *See* Boyer, *supra* note 104, at 598–619.

153. *See* Boyer, *supra* note 104, at 606.

required the explicit use of infinite sets.[154] In addition, other work in analysis naturally focused on various sets of numbers, many of which were infinite.[155] Thus many mathematicians, most notably Georg Cantor, were led to conceptualize and work with the actual infinite, and they felt that this could be done in such a way so as to deal once and for all with Zeno-type Paradoxes.[156] Unfortunately, the introduction generated more paradoxes.[157]

Language barriers may have prevented a full discussion of the issues raised by non-Euclidean geometry until the late 19th century, at which time the discussion was folded into discussions of the set-theoretic paradoxes.[158] The resulting turmoil is the current or third foundational crisis.[159] Three developments have emerged from this crisis.

First, mathematicians of the early 20th century attempted to deal with the lack of any focused philosophy of mathematics.[160] Most important for the purposes of this Article are the three competing approaches that appeared around the turn of the century and that still shape the contours of the philosophic discussion: Logicism, Intuitionism, and Formalism.

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154. In fact, such approaches involved infinite sets of natural numbers. *See* Davis & Hersh, *supra* note 34, at 331.

155. *See* Dauben, *supra* note 141, at 6–46.

156. *See* 2 Kramer, *supra* note 29, at 319.

157. *See id.* One of the easiest to understand is the Cantor Paradox that is described by Howard Eves and Carroll Newsom in non-technical terms as follows:

> In his theory of sets, Cantor had succeeded in proving that for any given [infinite] number there is always a greater [infinite] number, so that just as there is no greatest natural number, there also is no greatest [infinite] number. Now consider the set whose members are all possible sets. Surely no set can have more members than this set of all sets. But if this is the case, how can there be a[n] [infinite] number greater than the [infinite] number of this set?

Eves & Newsom, *supra* note 138, at 296–97.

It soon became clear that the Cantor Paradox is only one of a larger collection of troublesome "self-referential" paradoxes, some of which were known to antiquity. *See* Fraenkel et al., *supra* note 3, at 5–12; *see also* Evert W. Beth, *The Foundations of Mathematics* 481–94 (1959). Some scholars note that not all of these paradoxes are intrinsically related to problems concerning infinity. *See* G. Kreisel, *Two Notes on the Foundations of Set-Theory*, 23 Dialectica 93, 102 n.1 (1969).

158. *See* Hans Freudenthal, *The Main Trends in the Foundations of Geometry in the 19th Century*, in *Proceedings of the 1960 International Congress in Logic, Methodology and Philosophy of Science* 613, 616 (Ernest Nagel et al. eds., 1962).

159. *See* Beth, *supra* note 157, at 640–41. Some commentators restrict the current crisis to the set-theoretic paradoxes. *See* Eves & Newsom, *supra* note 138, at 296; Fraenkel et al., *supra* note 3, at 14.

160. *See* Michael Dummett, *Elements of Intuitionism* 1 (1977); Wilder, *supra* note 101, at 192.

The Logicist school believes that the correct philosophical tack lies in viewing mathematics as a part of logic.[161] Their program involves a reduction of all mathematical concepts, including the actual infinite, to purely logical notions in such a way that mathematical "truths" can be developed within logic without the appearance of paradoxes. This approach represents the confluence of two developments.[162] One development involved a reductive approach to mathematical concepts. In broad outline, this reduction began with the amalgamation of algebra and geometry to form analytic geometry, continued with the arithmetization of the limit concept, and reached fruition with descriptions of real numbers in terms of sets of natural numbers. The second development involved the search for a symbolic notation for the laws of logic.[163] Such a notation had been contemplated as early as Leibniz, but the first substantial success was obtained by George Boole in the middle of the 19th century. The link between these two developments was Gottlob Frege, who attempted to take reductionism one step further and analyze the notion of natural number in terms of more primitive logical notions and to develop a notation for a system that was far more detailed and general than Boole's relatively primitive system. Although Bertrand Russell discovered serious flaws in Frege's efforts,[164] his work had a

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161. For expositions of the Logicist position, see Körner, *supra* note 97, at 32–51; Rudolph Carnap, *The Logicist Foundations of Mathematics*, in *Philosophy of Mathematics* 41 (Paul Benacerraf & Hilary Putnam eds., 2d ed. 1982) [hereinafter Benacerraf & Putnam].

162. *See* Leon A. Henkin, *Are Logic and Mathematics Identical?*, in *The Chauvenet Papers: A Collection of Prize-Winning Expository Papers in Mathematics* 353, 354–55 (James C. Abbott ed., 1978).

163. For a general discussion of this development, see William Kneale & Martha Kneale, *The Development of Logic* 404–34 (1962).

164. *See* Letter from Bertrand Russell to Gottlob Frege (June 16, 1902), in *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, at 124 (Jean van Heijenoort ed., 1967) [hereinafter van Heijenoort] (describing so-called Russell Paradox). The reader should consider carefully Frege's response to Russell. *See* Letter from Gottlob Frege to Bertrand Russell (June 22, 1902), in van Heijenoort, *supra*, at 126. In Russell's words:

> As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.

Letter from Bertrand Russell to Jean van Heijenoort (Nov. 23, 1962), in van Heijenoort, *supra*, at 127.

substantial influence on the great early Logicist effort—Alfred North Whitehead and Russell's *Principia Mathematica*.[165]

There are a number of problems with the Logicist approach.[166] Even if the program can be carried out, it merely substitutes logical for mathematical questions, a fact illustrated by philosophical divisions within the Logicist camp itself.[167] The issue of mathematics as a science, for example, is dependent on a particular Logicist account of logic. A Logicist account of mathematics as a technology must somehow explain the relation between the empirical and the logical. Furthermore, Logicists always have had trouble explaining how their machinery for dealing with actual infinity is logical in nature at all. On the other hand, their early work did suggest the possibility of introducing the actual infinite in such a way as seemingly to avoid paradoxes.[168]

The Intuitionists trace their roots to Kant and believe that the correct approach is based on a mental faculty of intuition that is more basic than any logical, or even linguistic, ability.[169] According to this school, mathematics involves mental construction rather than the discovery of "truths." As Arend Heyting puts it, "[T]he intuitionist mathematician proposes to do mathematics as a natural function of his intellect, as a free, vital activity of thought. For him, mathematics is a production of the mind."[170] Stephan Körner, however, reminds us that "[t]he subject-matter of intuitionist mathematics . . . is intuited non-perceptual objects and constructions which are introspectively self-evident."[171] It is important to understand what the attitude described by Körner entails. Stephen Kleene explains the essential implication for the purposes of this Article as follows:

> The familiar mathematics . . . as developed prior to [the Intuitionist] critique or disregarding it, we call *classical*; the mathe-

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165. Alfred N. Whitehead & Bertrand Russell, *Principia Mathematica* (1st ed. 1910).

166. *See* Körner, *supra* note 97, at 52–71; *see also* Carnap, *supra* note 161.

167. *See* Beth, *supra* note 157, at 363–64; Fraenkel et al., *supra* note 3, at 335; Körner, *supra* note 97, at 34–38; Penelope Maddy, *Realism in Mathematics* 26–27 (1990).

168. In essence, one asserts that the universe of sets occurs in levels. A set at one level only has members from previous levels. Thus, Cantor's Paradox seemingly is avoided by precluding the set of all sets. For an intuitive description, see Herbert B. Enderton, *Elements of Set Theory* 7–9 (1977). *See also* Tiles, *supra* note 116, at 154–58.

169. For expositions of the Intuitionist position, see Fraenkel et al., *supra* note 3, at 210–74; Körner, *supra* note 97, at 119–34; Arend Heyting, *The Intuitionist Foundations of Mathematics*, *in* Benacerraf & Putnam, *supra* note 161, at 52–61.

170. Heyting, *supra* note 169, at 52.

171. Körner, *supra* note 97, at 120.

> matics . . . which [the Intuitionists] allow, we call *intuitionistic*. The classical includes parts which are intuitionistic and parts which are non-intuitionistic.
>
> The non-intuitionistic mathematics which culminated in the theories of [Cantor and others], and the intuitionistic mathematics . . . differ essentially in their view of the infinite. In the former the infinite is treated as *actual* or *completed* or *extended* or *existential*. An infinite set is regarded as existing as a completed totality, prior to or independently of any human process of generation or construction, and as though it could be spread out completely for our inspection. In the latter, the infinite is treated only as *potential* or *becoming* or *constructive*.[172]

But now, more than two millennia after the dilemma first arose, the implications of a narrower attitude towards infinity can be much more starkly described. Although the Intuitionist attitude apparently does dispose of the paradoxes, it leads this school to reject so much of the classical perspective that the result "has turned out to be considerably less powerful than classical mathematics, and in many ways . . . much more complicated to develop. . . . This is the fault found with the intuitionist approach—too much that is dear to most mathematicians is sacrificed."[173]

There are other problems with Intuitionism.[174] With respect to the certainty of mathematics as a science, the position is subject to the standard intersubjectivity problems of theories that analyze validation in terms of self-evident experiences. Moreover, the problems for mathematics as a technology raised by the separation of intuition and perception have been exposed more fully in an era of developments in physics not imagined in Kant's time. As with Logicism, the Intuitionist school embraces a range of positions.[175]

The heart of the Formalist position is an interest in formal deductive systems. Subsection 5 discusses this school in greater detail. For now, the reader should be aware of the following. One group of Formalists takes the position that mathematics is preeminently syntactic—merely an empty game of symbol manipulation. Others allow for more semantic

172. Stephen C. Kleene, *Introduction to Metamathematics* 48 (1952).

173. Eves & Newsom, *supra* note 138, at 304. For a more technical discussion of this point, see Kleene, *supra* note 172, at 46–53. *See also* Fraenkel et al., *supra* note 3, at 210–74.

174. For a general discussion of these problems, see Körner, *supra* note 97, at 135–55.

175. *See* Fraenkel et al., *supra* note 3, at 214–20.

content by asserting that certain of these systems also can be viewed in terms of a type of mathematical if-thenism that studies which mathematical conclusions follow semantically from given mathematical premises. Both of these positions differ widely from the Logicist and Intuitionist views. In its most mature form under Hilbert, however, Formalism holds out hope for a reconciliation with the other two schools. There are problems with each of these Formalist approaches. In particular, Hilbert's dreams have been dealt a serious, if not fatal, blow by the work of Gödel.

The second development to emerge from the current crisis is a new embracing of the axiomatic method.[176] This has occurred despite the failure of any of the three schools to provide a generally acceptable philosophy of mathematics.[177] At one level, the axiomatic method involves careful elaboration of the various branches of mathematics and their underlying logic.[178] The axiomatic method, however, also involves studying the resulting systems as objects *per se* in terms of properties such as consistency.[179] Thus, mathematicians once again have come to appreciate the importance of the axiomatic method both as a tool for doing mathematics[180] and as a prelude to dealing with foundational issues.[181] In this sense, mathematics has returned to its Greek origins.

The final development is a growing awareness of mathematics as an art. The notion of an elegant proof dates back at least to Aristotle,[182] but mathematicians now also see their systems in terms of works of art.[183]

Once again, this mathematical crisis does not exist in a vacuum. The past 150 years have seen a complex and comprehensive reevaluation of the Western intellectual tradition, and the current foundational crisis is an integral part of this larger intellectual context.[184]

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176. Even the Intuitionists, who are generally hostile to formalizations, have found axiomatics useful. *See* Beth, *supra* note 157, at 433–34; Dummett, *supra* note 160, at 300; Fraenkel et al., *supra* note 3, at 239–40.

177. *See* Davis & Hersh, *supra* note 34, at 346.

178. *See* Kline, *Mathematical, supra* note 29, at 1026–27.

179. *See id.*

180. *See* Wilder, *supra* note 101, at 101–02.

181. *See* 2 Kramer, *supra* note 29, at 444–45; Raymond L. Wilder, *Introduction to the Foundations of Mathematics* 278 (2d ed. 1965).

182. *See* Boyer, *supra* note 104, at 117.

183. *See* Sullivan, *supra* note 32.

184. *See* Purcell, *supra* note 51, at 47–73; Tiles, *supra* note 98, at 1–6, 166–74; Williams, *supra* note 12, at 432–69; *cf.* Stephen G. Simpson, *Partial Realizations of Hilbert's Program*, 53 J. Symbolic Logic 349, 358 (1988).

5. *Mathematical Interlude*

This subsection presents most of the mathematical details contained in this Article. The first part provides a brief discussion of non-Euclidean geometry, and the second part describes Formalism and Gödel's Theorems.

a. *Non-Euclidean Geometry*[185]

Practical geometry was known to civilizations pre-dating the Greeks. The Egyptians and Babylonians had developed solutions to a wide range of problems, but these solutions were obtained by a mixture of experimentation, guessing, analogy, and intuition. The Greeks were familiar with these results, but required that they be established by a type of deductive reasoning.

This Greek innovation has been called material axiomatics. Howard Eves describes it as follows:

> (A) Initial explanations of certain basic technical terms of the discourse are given, the intention being to suggest to the reader what is to be meant by these basic terms.
>
> (B) Certain primary statements concerning the basic terms, and which are felt to be acceptable as true on the basis of properties suggested by the initial explanations, are listed. These primary statements are called the *axioms*, or the *postulates*, of the discourse.
>
> (C) All other technical terms of the discourse are defined by means of previously introduced terms.
>
> (D) All other statements of the discourse are logically deduced from previously accepted or established statements. These derived statements are called the *theorems* of the discourse.[186]

Euclid's *Elements* is the archetype of this form of reasoning. The *Elements* contain several basic technical terms, including point, straight line, and plane surface. There are axioms or assumptions common to all mathematical reasoning, including the assumption that equals added to equals yield equals. There are postulates or assumptions specific to

185. The discussion in this section is based largely on Howard Eves, *A Survey of Geometry* 1–11, 282–88, 375–79 (rev. ed. 1972). Many details can be found in Marvin J. Greenberg, *Euclidean and Non-Euclidean Geometries: Development and History* (1974); Trudeau, *supra* note 113.

186. Eves, *supra* note 185, at 11.

geometric reasoning, including the assumption that two points determine one and only one straight line.[187] There are numerous derived terms such as right angle. Postulates mentioning derived terms are in essence postulates about the basic terms. Finally, the rules of deduction are not spelled out but are indicated by the methods of proof employed in the *Elements*.[188]

Parallelism caused trouble from the beginning. As has been described above, Greek mathematicians shunned direct appeals to the infinite.[189] As a result, assertions about parallelism would be suspect to the extent that they did not involve finite figures or finite parts of figures; that is, such statements would be problematic to the extent that they conceived of a line as an infinite whole.[190] On the other hand, many results seemed to require some assumptions about parallelism.[191]

Euclid proposed a framework emphasizing potential rather than actual infinity.[192] Euclid's definition of a "straight line" actually aims at what today would be called a line segment. The Second Postulate then asserts that straight lines can be extended indefinitely in either direction.[193] Parallelism is defined by saying that parallel straight lines are coplanar straight lines such that no extensions intersect.[194] The cornerstone of Euclid's framework is the famous Fifth Postulate:

> If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines are not parallel. In particular, there are extensions meeting on that side on which the angles are together less than two right angles.[195]

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187. On the varying usages of the terms axiom and postulate, see Howard Eves, *An Introduction to the History of Mathematics* 124–25 (4th ed. 1975).

188. A number of "gaps" in the *Elements* were discovered and fixed over time, but these gaps are not relevant for the purposes of this Article. *See* Greenberg, *supra* note 185, at 57.

189. *See supra* notes 116–18 and accompanying text.

190. *See* Kline, *Mathematical, supra* note 29, at 175.

191. *See* Trudeau, *supra* note 113, at 44–99.

192. *See* Flegg, *supra* note 118, at 256; Kline, *Mathematical, supra* note 29, at 175.

193. *See* Trudeau, *supra* note 113, at 30–32, 39–40.

194. *See id.* at 39.

195. Trudeau describes this as follows:

> In [the figure], *EF* is a straight line "falling" on two straight lines *AB* and *CD*. (Modern textbooks would call *EF* a "transversal.") There are two pairs of "interior angles on the same side": angles 1 and 2, and angles 3 and 4. Postulate 5 says that if either pair adds up to less than 180° then *AB* and *CD*, if extended far enough, will intersect on the same side of *EF*. Specifically, if [angles 1 and 2 add up to less than 180°] then *AB* and *CD* will meet to the right

Even this framework made Greek mathematicians uncomfortable. As an immediate matter, the Fifth Postulate deals only with potentially infinite figures.[196] Nonetheless, the Greeks were hesitant to accept as self evident such a statement about the potentially infinite.[197] This concern is understandable given the earlier problems Greeks had faced with the infinite.[198] As a result, mathematicians attempted to rework the framework.[199] Efforts focused on trying to replace Euclid's Fifth Postulate with something less objectionable or to derive it from the remaining assumptions.[200] None of these efforts was successful.[201]

In the 18th century, attempts to apply the method of proof by contradiction to derive the Fifth Postulate led not to contradictions but instead to a strange collection of what is now known to be the theorems of one type of non-Euclidean geometry.[202] The work of Girolamo Saccheri is typical.[203] He could show without the Fifth Postulate that if, in a quadrilateral *ABCD*, angles *A* and *B* are right angles and sides *AD*

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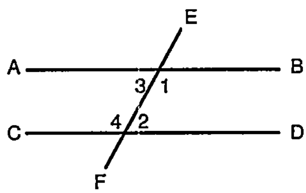
of *EF* . . . , and if [angles 3 and 4 add up to less than 180°] they will meet to the left. Before Euclid makes use of Postulate 5 he will prove that it is impossible for [the pairs of angles] to *both* be less than 180°.



Id. at 42.

196. The use of potential infinity in the Fifth and Second Postulates, however, did raise the question of whether physical space is infinite. *See* Kline, *Mathematical, supra* note 29, at 177.

197. *See id.*

198. *See supra* notes 109–10 and accompanying text.

199. Some went so far as to try to use a different definition of parallel. *See* 1 Heath, *supra* note 107, at 358.

200. *See* Greenberg, *supra* note 185, at 19. Some commentators find the artistic dimension of mathematics at work here as well. As far back as Aristotle, there was a notion that a proof would be more elegant if it used fewer or simpler assumptions. *See* Boyer, *supra* note 104, at 117. Also, the Fifth Postulate looked and felt more like a theorem than a postulate; thus, its elimination was a matter of aesthetics. *See* Trudeau, *supra* note 113, at 118; Wilder, *supra* note 101, at 9, 98–99.

201. *See* Roberto Bonola, *Non-Euclidean Geometry* 1–21 (H.S. Carslaw trans., 1911); Greenberg, *supra* note 185, at 19–21, 122–27; Trudeau, *supra* note 113, at 119–31.

202. *See* Bonola, *supra* note 201, at 22–51; Greenberg, *supra* note 185, at 127–29.

203. For a brief discussion of Saccheri, see Eves, *supra* note 185, at 284–85.

and *BC* are equal, then angles *D* and *C* are equal.[204] There are thus three possibilities: *D* and *C* are equal acute angles (measure less than 90 degrees), *D* and *C* are right angles, and *D* and *C* are equal obtuse angles (measure greater than 90 degrees). The second possibility implies (in fact is equivalent to) the Fifth Postulate, and thus Saccheri hoped to derive contradictions from the other two cases. He could do this easily for the third case, but not the first. Instead, he derived many of what we know now to be the theorems of one type of non-Euclidean geometry. Mathematicians, however, were unable to conceive of this interpretation of their work.[205] Saccheri, for example, tried to assert a contradiction based on vaguely described ideas of the nature of a line.[206]

It remained for the mathematicians of the early 19th century to realize that such results evidenced a non-Euclidean geometry.[207] Indeed, mathematicians were able to show that the consistency of this system followed from the consistency of Euclidean geometry. This result was established by providing within Euclidean geometry a model of non-Euclidean geometry in such a way that any inconsistency in non-Euclidean geometry could be translated into an inconsistency in Euclidean geometry.[208] Towards the end of the 19th century, mathematicians realized that the contradiction in Saccheri's third case could be removed by modifying some of Euclid's other assumptions. The

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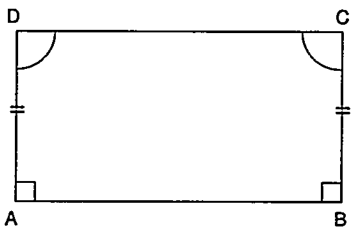
204.



205. *See* Bonola, *supra* note 201, at 43, 49–50; Greenberg, *supra* note 185, at 129. This inability might be explained by Kant's influence. *See* Bonola, *supra* note 201, at 64, 92–93, 121.

206. *See* Bonola, *supra* note 201, at 43; Greenberg, *supra* note 185, at 129; Trudeau, *supra* note 113, at 142.

207. *See* Bonola, *supra* note 201, at 64–113; Greenberg, *supra* note 185, at 131, 143–50; Trudeau, *supra* note 113, at 157–59.

The exact path from 18th century mathematicians like Saccheri to early 19th century mathematicians like Carl Friedrich Gauss, John Bolyai, and Nicholas Lobachevski is not easy to trace. For one attempt, see Bonola, *supra* note 201, at 66–113.

208. *See* Greenberg, *supra* note 185, at 181–84. In fact, one can produce a model of Euclidean geometry within non-Euclidean geometry so that the two geometries are "equiconsistent." *See id.* at 248.

resulting (different type of) non-Euclidean geometry is once again no less consistent than Euclidean geometry.[209]

Some idea of the difference in these geometries can be obtained by considering the so-called Euclidean Parallel Postulate version of Euclid's Fifth Postulate.[210] Take a straight line *l* and a point *P* not on *l* or any extension of *l*. The Euclidean Parallel Postulate states that there exists a straight line through *P* and parallel to *l*, and that there is "only one" in the sense that any two straight lines through *P* and parallel to *l* are collinear (i.e. lie in a common straight line).[211] In the presence of Euclid's other assumptions, the Euclidean Parallel Postulate is equivalent to Euclid's Fifth Postulate.[212] In fact, many readers may have been introduced to an axiomatization that used the Euclidean Parallel Postulate rather than Euclid's Fifth.[213] In Saccheri's first case, however, there exist straight lines through *P* and parallel to *l* that are not collinear. In fact, an axiomatization of this type of geometry can be obtained by replacing the Euclidean Parallel Postulate with this statement.[214] In the geometry resulting from Saccheri's third case, there are no straight lines through *P* and parallel to *l*.[215] As Saccheri's work shows, one cannot simply replace the Euclidean Parallel Postulate with such a statement; the result is an inconsistent system. Modifications must be made in some of the other assumptions as well.[216]

Some descriptions of non-Euclidean geometry in the legal literature are what can only be described as confused. One commentator, for example, tells us that "[n]on-Euclidean geometry postulates the intersection of parallel lines."[217]

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209. Once again, mathematicians constructed a model within Euclidean geometry. *See id.* at 275–80.

210. The postulate was popularized by John Playfair's 18th century presentation of Euclidean geometry, although the postulate itself is much older. *See id.* at 17.

211. *See id.*

212. *See id.*

213. One also sees the so-called Hilbert Parallel Postulate, which states that any two parallel straight lines through *P* and parallel to *l* are collinear. *See id.* at 84. The part of the Euclidean Parallel Postulate asserting the existence of a parallel straight line follows from the other assumptions. *See id.*

214. *See* Trudeau, *supra* note 113, at 159, 173, 177.

215. The elementary treatment sketched here can be significantly generalized. *See* H.S.M. Coxeter, *Non-Euclidean Geometry* (5th ed. 1957); Greenberg, *supra* note 185, at 280–88.

216. *See* Eves, *supra* note 185, at 287–88; Greenberg, *supra* note 185, at 275–80.

217. George R. Nock, *The Point of the Fourth Amendment and the Myth of Magisterial Discretion*, 23 Conn. L. Rev. 1, 2 n.9 (1990). *See also* Rudolph J. Peritz, *The Predicament of Antitrust Jurisprudence: Economics and the Monopolization of Price Discrimination Argument*, 1984 Duke L.J. 1205, 1251 n.267 ("[T]hey are all parallel yet they all meet."). For a better attempt,

Two of the most common misstatements in describing non-Euclidean geometry are: (1) one obtains Saccheri's third case geometry merely by replacing Euclid's Fifth Postulate with the assumption that "there are no parallels," and (2) the great circles on a sphere form a model of Saccheri's third-case geometry.[218] Legal scholars do not appear to be immune from such descriptive mistakes.[219]

It has been noted that if Euclidean geometry is consistent, then so is the type of non-Euclidean geometry resulting from Saccheri's first case. With a little work, it follows from this result that if Euclidean geometry is consistent, then Euclid's Fifth Postulate can neither be derived from, nor refuted by, the other assumptions.[220] That is, if Euclidean geometry is consistent, then these other assumptions form an "incomplete system" in the sense that Euclid's Fifth Postulate is "undecidable" with respect to these assumptions.[221] Thus, non-Euclidean geometry provided an early example of an incompleteness result. The results known as Gödel's Incompleteness Theorems therefore must derive their significance from something other than such an incompleteness *per se*. This significance is due to their context.

b. Gödel's Theorems

Providing a specific context for Gödel's Theorems is as important to understanding them as a description of the work itself.[222] An appropriate context can be developed from a number of perspectives. The choice here is to tell the story in terms of the mathematical Formalists.[223]

In outline, the story is as follows. The earliest incarnations of Formalism largely were self-contained competitors of the Logicist and Intuitionist schools. As such, there were a number of criticisms of these early Formalist approaches. More importantly, the competition among

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correct as far as it goes, see Burton M. Leiser, *Threats to Academic Freedom and Tenure*, 15 Pace L. Rev. 15, 60 n.247 (1994) ("Non-Euclidean geometries have been constructed on the premise that *no* such parallels can be drawn, and also on the premise that *more than one* such parallel can be drawn.").

218. For a discussion of why these are errors, see Greenberg, *supra* note 185, at 275–80.

219. For an example, see Peritz, *supra* note 217, at 1251 n.267.

220. *See* Greenberg, *supra* note 185, at 183–84.

221. These terms will be defined more precisely in the discussion of Gödel's Theorem immediately below.

222. *See* Dow, *supra* note 4, at 713.

223. The discussion is based on the views of Maddy, *supra* note 167, at 23–26. For a similar view, but one that is different in important respects, see Michael D. Resnik, *Frege and the Philosophy of Mathematics* 54–137 (1980).

the three schools eventually threatened to tear mathematics apart. Through his so-called Hilbert Program, David Hilbert reformulated Formalism in an attempt to deal with general critiques of Formalism and to unite the competing schools. This reformulation involves a careful balancing of the various positions. Unfortunately, Gödel's Theorems deal Hilbert's Program a serious, if not fatal, blow. In essence, Gödel's Theorems turn Hilbert's balancing against itself. This balancing is at the core of Gödel's Theorems. Indeed, the theorems themselves have specific hypotheses and conclusions and therefore are of limited scope and application. In particular, they do not apply to all formal systems. It is ironic that Gödel himself did not set out to destroy Hilbert's dreams. In fact, he was led to his results through attempts to carry out the Hilbert Program! Now for some details.

The core of the Formalist heritage is the study of formal as opposed to material systems. That is, in step (A) above[224] the basic terms are self-consciously viewed as undefined, and in step (B) the assumptions are self-consciously viewed as unjustified.[225] More specifically, a formal system consists of three parts: a formal language, a set of axioms, and a set of rules of inference.[226] The latter two comprise the deductive apparatus of the formal system. A formal language is given by specifying an alphabet (i.e. a particular collection of symbols) together with the collection of formulas over that alphabet (i.e. a particular collection of sequences of symbols). The axioms are given by specifying some subset of these formulas. In essence, rules of inference tell us that one formula, called the conclusion of the rule, can be inferred from certain other formulas, called the hypotheses of the rule. Given a formal system, one can define a notion of proof for that formal system. A proof is a sequence of formulas such that each formula is either an axiom or the conclusion of a rule of inference whose hypotheses precede the formula in the sequence. The last formula in a proof is called a theorem of the system or of the axioms, and we say that the proof is a proof of the theorem.[227] The reader must realize that a formal system is preeminently syntactic. The language of a formal system is not asserted to have any semantic content *per se*, although semantic considerations may have influenced the exact

224. *See supra* text accompanying note 186.

225. *See* Eves, *supra* note 185, at 338.

226. *See* Joseph R. Shoenfield, *Mathematical Logic* 1–5 (1967).

227. For the purposes of this Article, a formula is a finite sequence of symbols, a rule of inference has a finite number of hypotheses, and a proof is a finite sequence of formulas. An alphabet or a set of axioms can, however, be infinite, but not "too infinite." What "too infinite" means is beyond the scope of this Article.

form of the alphabet and formulas. The notion of consequence given by the deductive apparatus of a formal system similarly is preeminently syntactic. From this perspective, to say that a formula is a consequence of a set of hypotheses is to say in essence that there is a sequence of chicken scratches satisfying certain rules of syntactic manipulation.

A footnote illustrates these ideas in the context of a particular elementary formal system that can be called the basic propositional logic system.[228] Obviously, more complicated systems are utilized for more

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228. For *pedagogical reasons only*, the reader is encouraged to think of this system as having some semantic content: the semantics of "and," "or," "not," "if-then," and "if and only if." The formal system *qua* formal system, however, is preeminently syntactic and consists merely of the language, axioms, and rules of inference—even though, as in this example, their specifications may have been influenced by semantic considerations.

The alphabet of the language consists of three parts. First, an infinite set of propositional variables P_1, P_2, ... Second, a set of five propositional connectives: $\wedge$ (the formal counterpart of "and" or "conjunction"), $\vee$ (the formal counterpart of "or" or "disjunction"), $\neg$ (the formal counterpart of "not" or "negation"), $\rightarrow$ (the formal counterpart of "if-then" or "one-way implication"), $\leftrightarrow$ (the formal counterpart of "two-way implication" or "if and only if" or "iff"). Third, a set of two punctuation symbols (and). Before proceeding, note that the formal system is described from the outside; that is, it is described in terms of a so-called metatheory. The alphabet, for example, is just a set, but it is described from the outside (say in mathematical english) by listing its members.

The formulas are specified according to the following rules: (1) any propositional variable is a formula, and (2) if α and β are propositional formulas, then so are the following five: $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\neg\alpha)$, $(\alpha \rightarrow \beta)$, and $(\alpha \leftrightarrow \beta)$. For example, P_1 is a formula by (1). P_2 is a formula by (1). Hence $(P_1 \rightarrow P_2)$ is a formula by (2). (For pedagogical reasons, the reader might wish to think of this as saying, "If P_1, then P_2.") Hence $(\neg(P_1 \rightarrow P_2))$ is a formula by (2). (The reader might wish to think of this as saying, "It is not the case that P_1 implies P_2.") Note that the set of formulas has not been described by listing its members directly, but in terms of a so-called metatheoretical formation rule. Note the use of the metatheoretical symbols α and β to denote formulas of the formal system.

For any formulas α, β, γ, the following fourteen formulas are axioms. That is, each of (1)–(14) represents a so-called metatheoretical schema that provides a general template for the collection of the schema's instances. The set of axioms, like the set of formulas, is not described by listing its members. It is the collection of the instances of the schemas that is the set of axioms. For pedagogical purposes, the reader may want to think of these schemas as statements about the propositional connectives. For example, the reader may want to think of the first two as statements describing when "or" holds, and the third as a statement describing when "or" fails to hold. (For example, the reader might wish to think of (1) as saying "α implies (α or β).")

(1) $(\alpha \rightarrow (\alpha \vee \beta))$
(2) $(\beta \rightarrow (\alpha \vee \beta))$
(3) $((\neg\alpha) \rightarrow ((\neg\beta) \rightarrow (\neg(\alpha \vee \beta))))$
(4) $(\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta)))$
(5) $((\neg\alpha) \rightarrow (\neg(\alpha \wedge \beta)))$
(6) $((\neg\beta) \rightarrow (\neg(\alpha \wedge \beta)))$
(7) $(\alpha \rightarrow (\beta \rightarrow \alpha))$
(8) $((\neg\alpha) \rightarrow (\alpha \rightarrow \beta))$
(9) $(\alpha \rightarrow ((\neg\beta) \rightarrow (\neg(\alpha \rightarrow \beta))))$

complicated forms of reasoning, and the reader should pause now and examine this "elementary formal system" to gain some understanding of what it might mean to put law in a syntactic framework that can be described and analyzed mathematically.

For the early Formalists, a formal system depicted a part of mathematics as a game whose pieces are the formulas and whose moves are the rules of inference.[229] The object of this game is to produce proofs in the system; classification is largely syntactic in nature. Such a view, however, is subject to a number of criticisms.[230] With little room for semantics, this view is a total break with every preceding description of

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(10) $((\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \alpha) \rightarrow (\alpha \leftrightarrow \beta)))$
(11) $(\alpha \rightarrow ((\neg\beta) \rightarrow (\neg(\alpha \leftrightarrow \beta))))$
(12) $(\beta \rightarrow ((\neg\alpha) \rightarrow (\neg(\alpha \leftrightarrow \beta))))$
(13) $((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)))$
(14) $((\alpha \rightarrow \beta) \rightarrow (((\neg\alpha) \rightarrow \beta) \rightarrow \beta))$

What about the set of rules of inference? The propositional logic system has only one rule of inference—*modus ponens*. A metatheoretical template is given by saying that if α and β are any two formulas, then from the two hypotheses α and $(\alpha \rightarrow \beta)$, we may conclude β. This metatheoretical description completes the specification of the formal system. (Technically, *modus ponens* is a so-called 3-place relation on the set of formulas. *See* Elliott Mendelson, *Introduction to Mathematical Logic* 29 (1964).)

Let's consider an example of a proof. The following sequence of five formulas is a proof:

$((\neg P_1) \rightarrow (P_1 \rightarrow (\neg(\neg P_1))))$,
$((\neg(\neg P_1)) \rightarrow (P_1 \rightarrow (\neg(\neg P_1))))$,
$(((\neg P_1) \rightarrow (P_1 \rightarrow (\neg(\neg P_1)))) \rightarrow (((\neg(\neg P_1)) \rightarrow (P_1 \rightarrow (\neg(\neg P_1)))) \rightarrow (P_1 \rightarrow (\neg(\neg P_1)))))$,
$(((\neg(\neg P_1)) \rightarrow (P_1 \rightarrow (\neg(\neg P_1)))) \rightarrow (P_1 \rightarrow (\neg(\neg P_1))))$,
$(P_1 \rightarrow (\neg(\neg P_1)))$.

The first formula is an instance of schema (8) with P_1 in place of α and $(\neg(\neg P_1))$ in place of β. The second formula is an instance of (7) with $(\neg(\neg P_1))$ in place of α and P_1 in place of β. The third formula is an instance of (14) with $(\neg P_1)$ in place of α and $(P_1 \rightarrow (\neg(\neg P_1)))$ in place of β. The fourth formula results from an application of *modus ponens* to the first and third formulas with $((\neg P_1) \rightarrow (P_1 \rightarrow (\neg(\neg P_1))))$ in place of α and $(((\neg(\neg P_1)) \rightarrow (P_1 \rightarrow (\neg(\neg P_1)))) \rightarrow (P_1 \rightarrow (\neg(\neg P_1))))$ in place of β. The fifth formula results from an application of *modus ponens* to the second and fourth formula with $((\neg(\neg P_1)) \rightarrow (P_1 \rightarrow (\neg(\neg P_1))))$ in place of α and $(P_1 \rightarrow (\neg(\neg P_1)))$ in place of β. Thus, $(P_1 \rightarrow (\neg(\neg P_1)))$ is a theorem. The reader may wish to think of this as a proof of the statement "P_1 implies not(not(P_1))."

What about other systems of propositional logic? The system just described is so basic that the other propositional logic systems are obtained by adjoining a set of formulas Γ to the axioms of the basic propositional logic system. It is so basic that if α is a theorem in the resulting system, we say that α is a syntactic consequence of Γ even though the proof may use some of the basic system.

229. *See* Maddy, *supra* note 167, at 23.

230. For an overview of these criticisms, see Resnik, *supra* note 223, at 55–65.

mathematics as a science. In any case, the view is hardly an appealing conception of mathematics as a science. Moreover, Frege pointed out early on that the view poses serious problems for mathematics as a technology. What could such syntactic manipulations have to do with the success enjoyed by applied mathematics?[231] Finally, the view has little to offer in the way of coming to grips with the infinite.

For the next wave of Formalists, formal systems described mathematics as a species of if-thenism.[232] This view offers more room for semantics. Recall that a formal system is preeminently syntactic. How does one introduce semantics explicitly? Semantic content for a formal language can be given through an assignment of meaning to the alphabet and/or formulas of the language. Of course, different assignments might give different meanings. One also can develop a semantic notion of consequence. Roughly speaking, a formula is a semantic consequence of a set of hypotheses if the formula is true whenever the hypotheses are true. That is, a formula is a semantic consequence of a set of hypotheses if every assignment of meaning making the hypotheses true also makes the formula true. Critical for the if-thenist approach was the discovery of results relating the syntactic and semantic notions of consequence in specific settings. Once again, the reader is encouraged to look at a footnote developing these ideas in the context of the propositional logic system referred to above.[233]

231. *See* Maddy, *supra* note 167, at 23–24.

232. *See id.* at 25.

233. Consider the system for propositional logic described *supra* note 228. The connective and punctuation symbols are always given their intended meanings, so the choice comes in assigning the meanings to the propositional variables. A propositional variable is given a meaning by assigning to it the meaning *TRUE* or the meaning *FALSE*. An interpretation is an assignment of *TRUE* or *FALSE* to each propositional variable. Under an interpretation, every formula in the language is given meaning through the use of the meanings of the connectives—that is, through the truth tables for the propositional connectives. For example, in the interpretation in which all variables are assigned *TRUE*, the formula $(P_1 \vee P_2)$ has meaning *TRUE*. Consider a set Γ of propositional logic formulas. Let α be a formula. If α is true in every interpretation making all the formulas in Γ true, we say that α is a semantic consequence of Γ. Roughly speaking, to say that α is a semantic consequence of Γ is to say that α is true whenever Γ is. The key result is the so-called Propositional Completeness Theorem which says that α is a syntactic consequence of Γ if and only if it is a semantic consequence of Γ. For a general overview, see Geoffrey Hunter, *Metalogic: An Introduction to the Metatheory of Standard First Order Logic* 91–116 (1971). With such a result, there is more room for semantics in a formalist approach to propositional logic.

It must be admitted that until about 1930 most mathematicians tended to slide back and forth between syntactic and semantic statements with only a general understanding of the distinction and connection. *See* Gregory H. Moore, *Zermelo's Axiom of Choice: Its Origins, Development, and Influence* 256–57 (1982); R.L. Vaught, *Model Theory Before 1945*, in 25 Proc. Symp. Pure Mathematics 153, 160-61 (1974).

While there is more room for semantics under such a view, mathematics as a science is still at most the study of which mathematical conclusions follow semantically from given mathematical premises. This version of Formalism can find inspiration in some of the thoughts of Aristotle and Leibniz. Nonetheless, it also lacks a certain descriptive and normative appeal. Penelope Maddy puts it as follows:

> [W]hich . . . language is appropriate for the statement of premises and conclusions? . . . [F]rom among the vast range of arbitrary possibilities, why do mathematicians choose the particular axiom systems they do to study? [W]hat were historical mathematicians doing before their subjects were axiomatized? [W]hat are they doing when they propose new axioms?[234]

Moreover, there is still the issue of mathematics as a technology. Maddy describes the Frege problem for this version of Formalism as follows:

> The general thrust of the if-thenist's [account] seems to be that the antecedent of a mathematical if-then statement is treated as an idealization of some physical statement. The [technologist] then draws as a conclusion the physical statement that is the unidealization of the consequent.
>
> Notice that on this picture, the physical statements must be entirely mathematics-free; the only mathematics involved is that used in moving between them. . . . In other words [this account] requires that natural science be wholly non-mathematical, but it seems unlikely that science can be so purified.[235]

Finally, although if-thenism might offer a view of the consequences of adopting infinitary reasoning, it does not provide other means of dealing with the underlying arguments over its use.

Formalism assumed its most complex incarnation with David Hilbert. By any measure, Hilbert was one of the most important mathematicians of modern times. He obtained significant results in a variety of fields, and his famous list of twenty-three problems had a major influence on the development of 20th century mathematics.[236] His foundational stance

234. Maddy, *supra* note 167, at 25.

235. *Id.* at 25–26 (citations omitted).

236. For a translation of the problems, see David Hilbert, *Mathematical Problems*, 8 Bull. Am. Mathematical Soc'y 437 (1902).

has framed much of the subsequent debate. This Article is concerned with Hilbert the foundationalist.

The early Hilbert made several significant contributions to Formalism. Important for the purposes of this Article is his work on the consistency and syntactic completeness of formal systems.

There are a number of possible definitions of the consistency of a formal system. Perhaps the most natural for the framework of this Article is the statement that (1) there is a formula that is not a theorem. This statement makes it clear why inconsistent systems hold little interest. All formulas are theorems, so that the system offers no syntactic classificatory ability.[237] In the terminology of this Article, such a system holds little scientific interest. Perhaps the most intuitive formulation of consistency for the situations Hilbert proposed to consider is the statement that (2) there is no formula such that both the formula and its negation are theorems.[238]

As indicated above, 19th century mathematicians worked with material axiomatic systems.[239] Kleene describes the shortcomings in 19th century consistency arguments as follows:

> [The 19th century technique] was to give a "model." A *model* for [a material axiomatic theory] is simply a system of objects, chosen

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237. Given this, one might wonder what possibly could be meant by an assertion that an inconsistent system can be of real interest. For such an assertion, see Daniel J.H. Greenwood, *Beyond Dworkin's Dominions: Investments, Memberships, the Tree of Life, and the Abortion Question*, 72 Tex. L. Rev. 559, 576 (1994) (reviewing Ronald Dworkin, *Life's Dominion: An Argument About Abortion, Euthanasia, and Individual Freedom* (1993)).

238. That is, the formal systems of interest have a syntax of negation. For such systems, clearly (2) implies (1). The systems of interest also employ analogues of propositional schema (8) and *modus ponens*. *See supra* note 228. Using the schema and *modus ponens*, it is clear that the negation of (2) implies the negation of (1). For a general discussion of these two notions of consistency, see 1 Alonzo Church, *Introduction to Mathematical Logic* 108–09 (1956); Hunter, *supra* note 233, at 78–79.

One might also consider for each formula β the following statement: $(3)_\beta$ It is not the case that both β and $(\neg\beta)$ are theorems. For any formula β, clearly (2) implies $(3)_\beta$, and $(3)_\beta$ implies (1).

Although inconsistency initially was semantic in character, the approach described here is syntactic and "therefore applicable to a logistic system independently of the interpretation adopted for it." 1 Church, *supra*, at 108. *See also* Hunter, *supra*, at 78.

Indeed, this was why the Formalists adopted it! Thus, the following statement by Brown and Greenberg is problematic:

> It should be noted that it is meaningless to state that two propositions are inconsistent until one imposes an interpretation upon them. . . . It is only in light of a given symbolic interpretation that the notion of formal consistency has meaning.

Brown & Greenberg, *supra* note 4, at 1448 n.48 (citation omitted).

239. *See supra* text accompanying note 186.

> from some other theory and satisfying the axioms. That is, to each object or primitive notion of the axiomatic theory, an object or notion of the other theory is correlated, in such a way that the axioms become (or correspond to) theorems of the other theory. If this other theory is consistent, then the axiomatic theory must be. For suppose that, in the axiomatic theory, a contradiction were deducible from the axioms. Then, in the other theory, by corresponding inferences about the objects constituting the model, a contradiction would be deducible from the corresponding theorems.
>
> ....
>
> Consistency proofs by the method of a model are relative. The theory for which a model is set up is consistent, if that from which the model is taken is consistent.
>
> Only when the latter is unimpeachable does the model give us an absolute proof of consistency....
>
> For proving absolutely the consistency of classical [arithmetic], of analysis, and of set theory . . . , the method of a model offers no hope. No mathematical source is apparent for a model which would not merely take us back to one of the theories previously reduced by the method of a model to these.
>
> The impossibility of drawing upon the perceptual or physical world for a model [was also argued by Hilbert].[240]

Hilbert suggested a direct method that focuses on formal systems. Kleene describes Hilbert's idea as follows:

> This direct method is implicit in the meaning of consistency (at least as we now think of it), namely that no . . . contradiction (a proposition *A* and its negation *not A* both being theorems) can arise in the theory deduced from the axioms. Thus to prove the consistency of a theory directly, one should prove a proposition about the theory itself, i.e., specifically about all possible proofs of theorems in the theory. The mathematical theory whose consistency it is hoped to prove then becomes itself the object of a

240. Kleene, *supra* note 172, at 53–54.

> mathematical study, which [study] Hilbert calls "metamathematics" or "proof theory."[241]

Hilbert also was interested in the syntactic completeness of certain formal systems. "*Ignoramus et ignorabimus*"—we are ignorant and we shall remain ignorant. This was the catch phrase of Emil duBois-Reymond who, in asserting that certain problems (such as the nature of matter and force) were unsolvable in principle, represented a pessimistic assessment of the ultimate power of the human intellect.[242] As a mathematician, Hilbert found duBois-Reymond's position abhorrent.[243] In Hilbert's words:

> [The] conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no *ignorabimus*.[244]

Moreover, Hilbert had definite ideas about what this means:

> Occasionally it happens that we seek the solution under insufficient hypotheses . . . and for this reason do not succeed. The problem then arises: to show the impossibility of the solution under the given hypotheses . . . . [E]very definite mathematical problem must necessarily be susceptible of an exact settlement, either in the form of an actual answer to the question asked, or by the proof of the impossibility of its solution . . . .[245]

In contrast to duBois-Reymond's pessimism, Hilbert's beliefs were much more optimistic. A so-called mathematical sentence is a formula with the syntactic structure of a "definite mathematical problem."[246] Hilbert believed that given any formal system employed by mathematicians and any sentence of that system, mathematicians eventually would be able to determine that the sentence is undecidable (neither it nor its negation is provable) or determine what is decided. This belief was his general response to duBois-Reymond. It is clear from the quotation above that Hilbert did not expect all formal systems to be what is called syntactically complete (all sentences are decidable).

241. *Id.* at 55.

242. *See* Constance Reid, *Hilbert* 13 (1970).

243. *Id.*

244. Hilbert, *supra* note 236, at 445.

245. *Id.* at 444.

246. For a discussion of sentences in some of the systems Hilbert proposed to study, see *infra* note 278.

Nonetheless, the syntactic completeness of a particular system would be a type of specific response to duBois-Reymond's brand of pessimism in so far as that system is concerned. Moreover, a syntactically complete system has a nice mathematical property—it answers every definite question put to it, so that no more axioms need be considered.[247] If the system also is consistent, then its answers are consistent.

For the purposes of this Article, the early foundational Hilbert was an if-thenist[248] concerned with the consistency and syntactic completeness of various formal systems. By the 1920s, however, Hilbert could not maintain such a narrowly circumscribed position. In addition to the criticisms of Formalism described above, other developments forced him to posit what were in effect new and more sophisticated interpretations of his earlier concerns. The two previous foundational crises had spawned a variety of perspectives on mathematics. To mathematicians, however, these diverse approaches had seemed largely compatible with the then-existing mathematics. This was not the case with the third crisis. Arguments were becoming increasingly divisive,[249] and Hilbert believed

247. *See* Tiles, *supra* note 98, at 95. For many of the formal systems Hilbert proposed to study, syntactic completeness had another nice implication.

Hilbert not only believed in the "solvability of every mathematical problem" as discussed in the text, he also hoped that for certain systems the relevant determinations could be performed in some sort of uniform manner. This hope led him to consider the so-called decision problem for a formal system: the general problem of whether an arbitrary sentence of the system is a theorem of the system. For the purposes of this Article, a uniform solution to the decision problem can be thought of in terms of a certain type of oracle. When given any sentence, the oracle correctly answers "yes" if the sentence is a theorem and "no" if it isn't. By asking the oracle about any particular sentence and its negation, one can determine that the sentence is undecidable or determine what is decided.

In the time of the early Hilbert, such a problem could be imagined as being solved only by actually providing an algorithm for solving it. That is, the oracle must represent some sort of mechanical procedure. *See* Martin Davis, *Computability and Unsolvability* 102 (1958); *cf.* Hilbert, *supra* note 236, at 458. Such an oracle would provide a complete refutation of duBois-Reymond as far as that system is concerned. For many of the formal systems Hilbert proposed to study, syntactic completeness implies that the decision problem is algorithmically solvable. For a discussion of this implication, see Herbert B. Enderton, *Elements of Recursion Theory*, in *Handbook of Mathematical Logic* 527, 546–48 (Jon Barwise ed., 1977) [hereinafter *Handbook*]. It is not clear that Hilbert realized this implication, although some of his disciples apparently did. *See* Hao Wang, *Reflections on Kurt Gödel* 55 (1987).

Hilbert did realize that there is a particular formal system such that an algorithmic solution of its decision problem could be used to produce algorithmic solutions to the decision problems of a wide range of formal systems. Hilbert called the decision problem for this particular system the *Entscheidungsproblem*. Thus, he was interested in showing that the *Entscheidungsproblem* is algorithmically solvable. *See* Davis, *supra*, at 134. It isn't. *See infra* note 297.

248. *See* Maddy, *supra* note 167, at 25.

249. For a general discussion of their divisiveness, see Reid, *supra* note 242, at 148–57.

that they threatened to "chop up and mangle the science," and in doing so, "run the risk of losing a great part of our most valuable treasures!"[250]

These problems led Hilbert to sketch what he believed to be an approach that would continue the Formalist emphasis on the study of formal systems, preserve the existing mathematics as much as possible, and appeal to devotees of the other two schools, especially the Intuitionists. He would "eliminate once and for all the questions regarding the foundations of mathematics."[251] The exact nature and scope of his ideas were never entirely clear,[252] however, and the description chosen here is suitable for this Article.[253]

For present purposes, one can say that Hilbert kept in mind three types of systems. First, there are the informal systems of reasoning used by mathematicians in their daily work. Second, there are the formal systems that are the counterparts of the informal systems. Finally, there is the metatheory used to describe and establish Hilbert's approach. The word "meta" is used because this third type of theory generally would be talking *about* the two other types of systems.[254] For example, the metatheory would deal with statements concerning the consistency and syntactic completeness of a formal system.

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250. *See id.* at 155.

251. David Hilbert, *Foundations of Mathematics*, in van Heijenoort, *supra* note 164, at 464, 464.

252. *See* Tiles, *supra* note 98, at 118.

253. This discussion is based on Kleene, *supra* note 172, at 53–65; Stephen C. Kleene & Solomon Feferman, *Foundations of Mathematics*, in 11 *Encyclopaedia Britannica* 630 (15th ed. 1974); Georg Kreisel, *Hilbert's Programme*, in Benacerraf & Putnam, *supra* note 161, at 207. For other discussions of Hilbert's ideas from a mathematical perspective, see Paul Bernays, *David Hilbert*, in 4 *The Encyclopedia of Philosophy* 496 (1972); Charles Parsons, *Foundations of Mathematics*, in 5 *Encyclopedia of Philosophy* 188 (1972); Dag Prawitz, *Philosophical Aspects of Proof Theory*, in 1 *Contemporary Philosophy: A New Survey* 235 (1981); C. Smorynski, *The Incompleteness Theorems*, in *Handbook*, *supra* note 247, at 821. For discussions from a more philosophical perspective, see Michael Detlefsen, *Hilbert's Program* (1986); Resnik, *supra* note 223, at 76–107; Tiles, *supra* note 98, at 89–128.

254. What things would be included in such a metatheory? As indicated above, Hilbert's consideration of the deductive aspects of formal systems led to the development of what he called metamathematics or proof theory. *See* text accompanying *supra* note 241. Mathematicians also were interested in the idea of algorithmic computability, and this interest eventually led to the development of recursion theory. *See supra* note 247; *infra* note 297. The interest in the semantics underlying informal systems led to the development of model theory—the study of the semantics of formal languages. *See* Chen Chung Chang & H. Jerome Keisler, *Model Theory* 1–4 (3d ed. 1990). Elaborations of Cantor's work with sets eventually led to set theory. *See infra* note 394. These four areas comprise what is commonly referred to as mathematical logic. There are many overlaps. For a detailed overview, see *Handbook*, *supra* note 253. For the purposes of this Article, one can more or less identify the metatheory with mathematical logic.

As indicated in the Kleene quotation above,[255] Hilbert was interested in various classical informal systems.[256] Hilbert distinguished two parts of these systems: real and ideal. Although Hilbert was not entirely clear on his definitions of these parts,[257] one may say for the purposes of this Article that the real portion was meant to correspond in some sense to the Intuitionistically acceptable portion and the ideal portion to the remainder.[258] For example, statements treating the infinite as actual are ideal. Though a Formalist, Hilbert was willing to agree with the Logicists and the Intuitionists that the real part was meaningful. He could not accept, however, the Logicist position that this meaning came from a reduction of mathematics to logic. He believed that logic and mathematics had to be developed jointly. If anything, Hilbert's position on the meaningfulness of real mathematics was much closer to the Intuitionists.[259]

The biggest challenge for Hilbert was with respect to the ideal part because the Logicists embraced the actual infinite and the Intuitionists rejected it. Like the Logicists, Hilbert was unwilling to jettison infinitary reasoning: "No one will drive us out of this paradise that Cantor has created for us."[260] On the other hand, he was willing to agree with the Intuitionists that statements about the actual infinite were not meaningful.[261] Kleene puts it as follows:

> The delicate point in [Hilbert's] position is to explain how the nonintuitionistic classical mathematics is significant, after having initially agreed with the intuitionists that its theorems lack a real meaning in terms of which they are true.[262]

To deal with this delicate point, Hilbert made both philosophical and mathematical appeals to the Intuitionists. Philosophically, Hilbert echoed some of Kant's ideas on the infinite,[263] thus invoking the intellectual

255. *See supra* text accompanying note 240.

256. For the connotation of the word "classical," see *supra* text accompanying note 172.

257. *See* Smorynski, *supra* note 253, at 823.

258. *See* Kleene, *supra* note 172, at 55.

259. *See* Tiles, *supra* note 98, at 104–05, 155.

260. Reid, *supra* note 242, at 177. Hilbert also said that a rejection "would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists. To prohibit [infinitary reasoning] is tantamount to relinquishing . . . mathematics altogether." Hilbert, *supra* note 251, at 476. For a discussion of what this latter quotation means in the context of arithmetic, see Tiles, *supra* note 98, at 104–07.

261. *See* Hilbert, *supra* note 251, at 470; *see also* Kleene, *supra* note 172, at 57.

262. Kleene, *supra* note 172, at 57.

263. *See supra* text accompanying note 147.

roots of Intuitionism itself.[264] His mathematical appeal—the so-called Hilbert Program—was more subtle and complex.

In the first step of his Program, Hilbert hoped to provide formal systems that captured both the real and ideal parts of each of the classical informal systems he proposed to study.[265] This capturing would preserve existing mathematics, and it also would appeal to the Logicists. Rudolph Carnap puts it as follows:

> [L]ogicism has a methodological affinity with formalism. Logicism proposes to construct the logical-mathematical system in such a way that, although the axioms and rules of inference are chosen with an interpretation of the primitive symbols in mind, nevertheless, *inside the system* the chains of deductions and of definitions are carried through formally as in a pure calculus, i.e., without references to the meaning of the primitive symbols.[266]

In the second step, Hilbert hoped to convince the Intuitionists that infinitary reasoning is "conservative"—that real statements produced with ideal reasoning can be produced with real reasoning alone. Infinitary reasoning therefore could be justified as purely instrumental.[267] In essence, this would provide a unitary version of the Greek bifurcated approach to the infinite. He would accomplish this by providing so-called finitistic metatheoretical arguments that the formal systems resulting from his first step are consistent.[268] Hilbert argued that this traditional Formalist consistency goal, if implemented through suitable (i.e. finitistic) means, would establish the conservation goal.[269] Moreover, Maddy argues that Hilbert's Program might simplify the problem of mathematics as a technology by treating infinitary reasoning as a justified heuristic.[270]

Although such a program may sound both distressingly vague and hopelessly ambitious, there were signs that it could be implemented. Some work showed that formal systems could capture various informal

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264. David Hilbert, *On the Infinite*, *in* van Heijenoort, *supra* note 164, at 367, 392. This appeal was never fully developed. For a modern attempt to do so, see Tiles, *supra* note 98, at 129–74.

265. *See* Kleene, *supra* note 172, at 53.

266. Carnap, *supra* note 161, at 41, 52.

267. *See* Maddy, *supra* note 167, at 24.

268. These arguments were to be metamathematical or proof-theoretic arguments of a type that are not too complex. *See* Kleene, *supra* note 172, at 59–65.

269. *See* Hilbert, *supra* note 251, at 474; *see also* Tiles, *supra* note 98, at 104–07; Prawitz, *supra* note 253, at 258; Smorynski, *supra* note 253, at 823–25, 846.

270. *See* Maddy, *supra* note 167, at 24.

systems.[271] Other work indicated that the requisite arguments for consistency might be found.[272] Moreover, certain Intuitionists held out hope that some accommodation was possible if Hilbert's ideas could be implemented.[273]

Unfortunately, the metatheoretical results known as Gödel's Incompleteness Theorems[274] indicate serious, if not insurmountable, difficulties for Hilbert's dreams. There are a variety of results encompassed by each of these theorems, and the versions chosen here are appropriate for the purposes of this Article.[275] Gödel's so-called First Incompleteness Theorem raises questions about the first step in Hilbert's Program.[276]

To understand the First Theorem, consider how one might try to capture a sophisticated system of classical informal reasoning with a formal system in such a way that the formal system is amenable to a suitable metatheoretical analysis. For the purposes of this Article, an appropriate formal system has several characteristics.

Certainly, the language should be simple yet have enough expressive power. That is, the syntax provided by the alphabet and formulas should be easy to analyze yet be capable of encompassing the requisite classical expression under the appropriate semantic interpretations As a footnote example for the ordinary high school arithmetic of the natural numbers

271. *See* Hunter, *supra* note 233, at 259–60 (listing some early capturing results); Wang, *supra* note 247, at 55–56.

272. *See* Wang, *supra* note 247, at 54; Kleene & Feferman, *supra* note 253, at 636; Wilfried Sieg, *Hilbert's Program Sixty Years Later*, 53 J. Symbolic Logic 338, 342 (1988).

273. *See* C. Smorynski, *Self-Reference and Modal Logic* 1 (1985).

274. Gödel's 1931 paper was originally published in German. For a translation approved by Gödel himself, see Kurt Gödel, *On Formally Undecidable Propositions of Principia Mathematica and Related Systems I*, *in* van Heijenoort, *supra* note 164, at 596.

At this point, comments on some of the more common sources cited by legal scholars are in order. The traditional lay introduction to Gödel's work is Ernest Nagel & James R. Newman, *Gödel's Proof* (1958). This book has been much praised since its publication, but it has its critics. *See* John Myhill, Book Review, 58 J. Phil. 209 (1961); Hilary Putnam, Book Review, 27 Phil. Sci. 205 (1960). Another commonly cited source is the Pulitzer Prize-winning Hofstadter, *supra* note 11. Reviews of this book tended to run to the extremes. *See* H.H. Pattee, Book Review, Int'l Stud. Phil., Spring 1983, at 87, 87. In any case, I would not recommend it for a focused analysis of Gödel's results and their context. There also are citations to Morris Kline, *Mathematics: The Loss of Certainty* (1980). This book must be evaluated in the light of some penetrating critiques. For one such critique, see J. Corcoran, Book Review, Mathematical Reviews 82e:03013 (1982).

275. For example, they encompass the versions discussed by legal scholars.

276. *See* Kleene & Feferman, *supra* note 253, at 637.

indicates, this is not a major obstacle for the situations Hilbert proposed to consider.[277]

In addition, one would want to include enough axioms and rules of inference so that the resulting formal system satisfies two properties. First, the system should be what is known as sound with respect to the intended (i.e. classical) semantics; that is, all provable sentences[278] are

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277. This footnote considers a language appropriate for attempts to formalize the ordinary high school arithmetic of the natural numbers.

The alphabet includes punctuation symbols, variable symbols, and propositional connective symbols, as well as operation symbols (such as + for addition), constant symbols (such as 0 for zero), and relation symbols (such as < for less than). Moreover, to express general statements of the type "there exists something with property P," it includes "quantification" symbols (such as $\exists$ for "there exists"). More formally, the alphabet contains the following: (1) a collection of variable symbols x_1, x_2, . . .; (2) the five propositional connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, and $\leftrightarrow$; (3) the quantifiers $\exists$ (the formal counterpart of "there exists" or "existential quantification"), and $\forall$ (the formal counterpart of "for all" or "universal quantification"); (4) the arithmetic function symbols + (the formal counterpart of "addition"), * (the formal counterpart of "multiplication"), and s (the formal counterpart of "successor"—the successor function applied to a natural number yields that number's successor); (5) the constant symbol 0 (the formal counterpart of "zero"); (6) the arithmetic relation symbols < (the formal counterpart of "is less than") and = (the formal counterpart of "is equal to"); and (7) two punctuation symbols (and).

The rules for forming formulas should correspond to the rules of mathematical syntax taught in high school. More formally, one first describes the rules for forming "terms," which are the formal counterparts of "arithmetic expressions." The terms are specified by the following rules: (1) 0 is a term; (2) any variable is a term; and (3) if τ_1 and τ_2 are terms, then so are $s(\tau_1)$, $(\tau_1 + \tau_2)$, and $(\tau_1 * \tau_2)$. For example, 0 is a term by (1). (0 is often called the numeral for zero.) Hence $s(0)$ is a term by (3). ($s(0)$ is in fact the numeral for the number one. Similarly, $s(s(0))$ is the numeral for the number two, and so on.) Hence $(s(0) + x_1)$ is a term by (3).

The formulas are then specified by the following rules: (1) if τ_1 and τ_2 are any two terms then $(\tau_1 = \tau_2)$ and $(\tau_1 < \tau_2)$ are formulas; (2) if α and β are any two formulas then so are $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\neg\alpha)$, $(\alpha \rightarrow \beta)$, and $(\alpha \leftrightarrow \beta)$; and (3) if α is any formula and x_i is any variable, then $(\exists x_i \alpha)$ and $(\forall x_i \alpha)$ are formulas. For example, x_1 is a term and 0 is a term, so $(0 < x_1)$ is a formula by (1). So $(\exists x_1(0 < x_1))$ is a formula by (3). Similarly, $(\exists x_1(x_1 < 0))$ is a formula.

Semantic content can be given to a formula in the language described above by interpreting the symbols in the classical manner. That is, + is interpreted as addition, etc. A full discussion of semantics is beyond the scope of this Article, but for some more details, see *infra* note 278.

278. This footnote continues the arithmetic discussion. Sentences have the form of a definite mathematical proposition. For the purposes of Hilbert's first step, non-variable symbols would have their intended classical informal interpretations. But what about the variable symbols? This is where sentences become important. In a sentence, all instances of variables are modified by quantifiers. For example, $(\exists x_1(0 < x_1))$ ("There is something greater than zero.") is a sentence, but $(0 < x_1)$ ("Zero is less than x_1.") is not a sentence. What is the importance of this distinction? Since all variables in a sentence already are explained by their quantifiers, the sentence's semantic content is determined as it stands, given the aforementioned interpretations of the non-variable symbols and the range of possible values for the variables. For other formulas, we will have to go further in general and assign values to the variables. For example, the sentence $(\exists x_1(0 < x_1))$ is true or false as it stands, but the semantic content of $(0 < x_1)$ requires more information about the value of x_1. In this sense, sentences have the syntactic structure of a definite mathematical statement.

(classically) true. Second, the system should be what is known as semantically complete with respect to the intended semantics; that is, all true sentences are provable. Now Hilbert's original concern about syntactic completeness (all sentences are decidable) takes on a new importance because for the situations he proposed to consider: (1) a sound, syntactically complete system is semantically complete,[279] and (2) a syntactically incomplete system is semantically incomplete.[280]

A sound, semantically complete formal system for a classical informal system can be easily obtained by taking the axioms to be the true sentences. One doesn't need any rule of inference! Such a system, however, might well be too complex to analyze metatheoretically in what Hilbert felt to be a suitable manner for his conservation goal. This indicates that there must be a proper balance for the collections of axioms and rules of inference. On the one hand, they should be rich enough to capture the informal reasoning. On the other hand, the collections should be simple enough so that the resulting formal system is amenable to an acceptable metatheoretical analysis. It turns out that an appropriate set of rules of inference is not difficult to delineate.[281] Moreover, the axioms appropriate for the type of formal system sketched here will have some technical (sometimes called logical) axioms.[282] These rules of inference and logical axioms are simple. In particular, the set of logical axioms is what is called recursive: simple enough that the question whether an arbitrary formula is a logical axiom can be answered algorithmically. Given this and the fact that an appropriate language is

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More formally, a sentence is a formula in which there are no so-called free variables. What is a free variable? Any variable occurring in a formula of type (1) is free. A variable is free in a formula of type (2) if it is free in α or β. A variable is free in a formula of type (3) if it is free in α and is not x_i.

279. Take any true sentence α. By syntactic completeness, either α or its negation is provable. But the negation, which is false, can't be provable because the system is sound.

280. A syntactically incomplete system is semantically incomplete since the informal system embraces the law of the excluded middle. *See* A.G. Hamilton, *Logic for Mathematicians* 119 (rev. ed. 1988). For the purposes of this Article, the law states that "*for every proposition A, either A or not A.*" Kleene, *supra* note 172, at 47.

281. The rules of inference appropriate for the type of formal system sketched here should certainly include *modus ponens*. In fact, it turns out that there are several other more technical rules of inference that would be included because of the use of quantifiers. A discussion of these rules is beyond the scope of this Article. (For the purposes of this Article, quantification takes place over individuals, as indicated *supra* note 277, not over relations or functions. That is, we are considering so-called first-order systems. For more on the implications of the phrase "first-order," see *infra* note 413.)

282. A discussion of the set of logical axioms is beyond the scope of this Article, but the set would include, for example, the instances of the schemas described *supra* note 228 for propositional logic. There would be others because of the use of quantifiers.

not a problem, the main balancing at issue in Hilbert's Program comes in the choice of the so-called non-logical axioms.

Gödel's First Incompleteness Theorem indicates the difficulties in balancing the needs for a simple yet rich set of non-logical axioms. Roughly speaking, what Gödel's result says is the following: Let *S* be any consistent formal system with a language, logical axioms, and rules of inference of the type indicated above that has a collection of non-logical axioms that is (1) rich enough to contain the formal counterparts of certain elementary arithmetic assertions about the natural numbers, and (2) simple enough to be recursive.[283] Then *S* is syntactically incomplete. More can be done, however. Given *S*, one can explicitly produce an undecidable arithmetic sentence.[284]

This result casts doubt on the first step of Hilbert's Program because it follows as a corollary, indeed it is often made part of the statement of Gödel's First Theorem, that such an *S* is semantically incomplete with respect to classical arithmetic![285]

On seeing Gödel's First Theorem for the first time, many readers say the following: "OK, so there's an undecidable sentence. Just add it (or its negation) into *S* as one of the non-logical axioms. Now that sentence is decidable; in fact, it is trivially provable." It is the case that the sentence is provable in the newly created system. However, this new system also is subject to Gödel's First Theorem so that there now is a sentence

283. The non-logical axioms must be rich enough to include, for example, the formal counterparts of various assertions about the properties of addition, multiplication, etc. A full elaboration is beyond the scope of this Article.

284. Note that if Θ is an undecidable sentence with respect to this *S*, then so is $(\Theta \wedge \alpha)$, where α is any sentence that is a theorem of *S*. (This is because Θ is a theorem if $(\Theta \wedge \alpha)$ is, and $(\neg\Theta)$ is a theorem if $(\neg(\Theta \wedge \alpha))$ is.) Given this and the fact that an undecidable sentence can be explicitly produced given *S*, the following statement is problematic:

[M]athematical undecidables are not easy to find....

> . . . Theoretically it follows from Godel's proof that there are an infinite number of undecidable mathematical statements, hard though they may be to discover. . . . Legal undecidables are demonstrably denser with respect to all the legal propositions we know than discovered mathematical undecidables are dense with respect to all the mathematical theorems that we know.

Anthony D'Amato, *Pragmatic Indeterminacy*, 85 Nw. U. L. Rev. 148, 173 n.80 (1990).

285. Because the various versions of the First Theorem cover the system developed in the Logicist *Principia Mathematica*, the result casts grave doubts on this approach to Logicism as well. *See* Tiles, *supra* note 98, at 116; I. Grattan-Guinness, *On the Development of Logics Between the Two World Wars*, 88 Am. Mathematical Monthly 495, 497–98 (1981); Henkin, *supra* note 162, at 356.

undecidable with respect to this new system![286] The reader may then wonder if one can't iterate some type of addition process to avoid the syntactic incompleteness while at the same time keeping consistency. Yes! Indeed, standard proofs of elementary versions of the so-called Lindenbaum Lemma establish the existence of consistent, syntactically complete extensions of consistent systems such as *S* through a type of iterated addition process.[287] This iterated addition, however, comes at a price. As the statement of Gödel's First Theorem indicates, to keep consistency one must have thrown in so much that the set of non-logical axioms is no longer recursive—roughly speaking, one must have thrown in so much that one can no longer tell by algorithmic means whether an arbitrary formula is a non-logical axiom![288]

Gödel's so-called Second Incompleteness Theorem raises questions about the second step in Hilbert's Program.[289] Roughly speaking, what it says is the following: Let *S* be any consistent formal system with a language, logical axioms, and rules of inference of the type indicated above that has a collection of non-logical axioms that is: (1) rich enough to contain the formal counterparts of certain elementary arithmetic assertions about the natural numbers, and (2) recursive.[290] Then *S* does not prove the formal arithmetic counterpart of a certain natural (i.e. of the type Hilbert envisioned)[291] statement of its own consistency. That is, as a matter of metatheoretical interpretation, the counterpart says that *S* is consistent. As a matter of classical interpretation, the sentence is merely some complicated arithmetic statement about the natural numbers.

This result casts doubt on the second step of Hilbert's Program because to the extent that the types of metatheoretical arguments Hilbert envisioned for these natural statements of consistency are encompassed

286. With some work, one can show that the new system *S'* is consistent. That is, adding an undecidable sentence to a consistent system does not affect consistency. *See* Mendelson, *supra* note 228, at 63. Moreover, *S'* will be rich enough if *S* is. Finally, the addition of a single axiom does not affect the recursivity of the set of non-logical axioms.

287. *See* Mendelson, *supra* note 228, at 64–65.

288. For a discussion of this sort of problem with the Lindenbaum Lemma, see *id.*

289. *See* Kleene & Feferman, *supra* note 253, at 638.

290. Further elaboration is beyond the scope of this Article, but the set of non-logical axioms would be more extensive than that required for the First Theorem.

291. For the importance of the qualifier "natural," see *infra* text accompanying notes 324–27.

by the types of formal systems described, the result indicates the impossibility of obtaining such arguments. As one mathematician puts it, "[Gödel] had proven two theorems which were then considered moderately devastating and which still induce nightmares among the infirm."[292]

Having indicated the balancing that led to the systems Gödel examined, it is useful to give some idea of techniques that can be used to obtain these results. Hilbert's Program involves carefully balanced consistent formal systems. On the one hand, the systems should be rich enough to capture sophisticated informal systems. On the other hand, the systems should be simple enough to be amenable to an acceptable metatheoretical analysis. In essence, what the techniques described here do is to turn this balancing against itself.

The key insight is that the elementary arithmetic assumptions (*EAA*) portion of the axioms of our system *S* is able to encode much of what is external to *S*.[293] For the purposes of the discussion of Gödel's First Theorem,[294] such an encoding has two essential features.

The first essential feature of the encoding is that in some sense *EAA* allows certain arithmetic sentences to refer to themselves.[295] This self-referencing feature of encoding essentially is due to the fact that the elementary arithmetic portion is rich enough to contain the formal counterparts of a good deal of informal arithmetic reasoning.

Second, *EAA* in some sense can accurately check purported proofs in *S*.[296] This feature of the encoding ability essentially is due to the fact that

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292. Smorynski, *supra* note 253, at 825.

293. The encoding has as its heart the so-called Gödel numbering technique, in which numbers are assigned to (sequences of) formulas.

294. There are several approaches to the proofs of Gödel's Theorems. This approach is based on J.N. Crossley et al., *What Is Mathematical Logic?* (1972); Smorynski, *supra* note 253, at 825–41. For another approach, see *id.* at 860–64.

295. More specifically, let $P(x_1)$ be an arithmetic formula. This notation is meant to indicate that the formula P really is a formula only about x_1—that is, x_1 is the only free variable. *See supra* note 278. One can show that there is an arithmetic sentence α such that (the system whose non-logical axioms are the) *EAA* proves that α is equivalent to $P([\alpha])$, where $[\alpha]$ is the numeral denoting the Gödel number of α, and $[\alpha]$ has been "substituted" for x_1 in $P(x_1)$. (Numerals are described more fully *supra* note 277. The definition of substitution is beyond the scope of this Article.) That is, $(\alpha \leftrightarrow P([\alpha]))$ is a theorem of *EAA*. In this sense, α is a sentence that refers to itself in terms of the formula $P(x_1)$.

296. More specifically, there is an arithmetic formula $Proofs_S(x_1,x_2)$ (thought of as saying "x_2 is a proof in this system S of x_1") such that for any numbers a and p (with numerals a and p), (1) if a is the Gödel number of a formula α and p is the Gödel number of a proof of α in S, then the formula $Proofs_S$(a,p) is a theorem of *EAA*—from which it will follow by one of the logical axioms dealing

(1) the collections of axioms and rules of inference of S are simple enough that purported proofs can be easily checked,[297] and (2) *EAA* is rich enough to reflect such checking.

In rough outline, one can establish Gödel's First Theorem as follows. Using the encoding ability, one can produce an arithmetic sentence, here denoted by π, that can be thought of as saying, "I am unprovable in the system S."[298] That is, as a matter of metatheoretical interpretation, the sentence says that the sentence is unprovable in S. As a matter of classical interpretation, the sentence is merely some complicated arithmetic statement. This π is the so-called Gödel sentence that Gödel developed by considering the ancient Liar-type Paradoxes.[299] Gödel metatheoretically used the consistency of S to show that π is not a theorem of S.[300] To show that the negation of π is not a theorem of S,

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with quantification that $(\exists x_2 Proof_S(a,x_2))$ is a theorem of *EAA*; (2) otherwise, $(\neg Proof_S(a,p))$ is a theorem of *EAA*.

In essence, this formula is constructed from a number of other arithmetic formulas encoding various syntactic concepts. For example, there is a formula $Form_S(x_1)$ (thought of as saying "x_1 is a formula in the language of S") such that for any number a (with numeral a), (1) if a is the Gödel number of a formula α then the formula $Form_S(a)$ is a theorem of *EAA*; (2) otherwise, $(\neg Form_S(a))$ is a theorem of *EAA*.

297. Gödel's original simplicity requirement was in fact more stringent than that described in the text. His formulation of the requirement helped stimulate sustained and systematic investigation of the concept of "algorithmically computable." *See* Stephen C. Kleene, *Origins of Recursive Function Theory*, 3 Annals Hist. Computing 52, 52–53 (1981).

This concept is important, and it is worth providing some discussion. The dream that reasoning can be reduced to some kind of calculation has influenced much of Western thought in general and mathematical thought in particular. *See* Hubert L. Dreyfus, *What Computers Can't Do: The Lifts of Artificial Intelligence* 67–87 (rev. ed. 1979) (discussing influence on Western thought); Hans Hermes, *Enumerability, Decidability, Computability* 26–30 (2d rev. ed. 1969) (discussing influence on mathematical thought). Indeed, Hilbert was interested in the algorithmic solvability of his so-called *Entscheidungsproblem*. *See supra* note 247. During the first half of the 1930s, mathematicians settled on a notion of recursive that is intended to be the mathematical counterpart of the concept of algorithmically computable. The (non-mathematical) assertion that the mathematical notion captures this concept is the so-called Church-Turing-Kleene-Post Thesis. *See* Hartley Rogers, Jr., *Theory of Recursive Functions and Effective Computability* 1–21 (1967); *see also* Kleene, *supra*, at 59.

As this work on recursivity was progressing, mathematicians realized that Gödel's original requirement could be relaxed to that described in the text. *See* Stephen C. Kleene, *The Work of Kurt Gödel*, 41 J. Symbolic Logic 761, 769 (1976). Moreover, Alonzo Church (and independently Alan Turing) showed that Hilbert's *Entscheidungsproblem* was not recursively solvable by showing that in certain formal arithmetic systems, the set of theorems is not recursive. Alonzo Church, *A Note on the Entscheidungsproblem*, 1 J. Symbolic Logic 40 (1936); A.M. Turing, *On Computable Numbers, with an Application to the Entscheidungsproblem*, 42 Proc. London Mathematical Soc'y 230 (1937).

298. Let $P(x_1)$ be the formula $(\neg(\exists x_2 Proof_S(x_1,x_2)))$. Then the corresponding self-referencing sentence can be thought of as saying, "I am unprovable in the system S."

299. *See* Gödel, *supra* note 274, at 598.

300. For ease of notation, let $Prov_S(x_1)$ denote $(\exists x_2 Proof_S(x_1,x_2))$. That is, $Prov_S(x_1)$ can be thought of as saying, "x_1 is provable in the system S." Suppose π were a theorem of S. It follows from (1) of

however, he needed to strengthen the consistency requirement to what is called ω-consistency.[301] This strengthening to ω-consistency is necessary, otherwise the system might prove the negation of the Gödel sentence.[302] By using a slightly different arithmetic sentence, however, J.B. Rosser was able to relax the requirement back to consistency.[303]

It is worth making one further comment on Gödel's original paper and the First Theorem. As noted above, arithmetical syntactic incompleteness for S implies arithmetical semantic incompleteness.[304] Gödel himself, however, was able to obtain arithmetical syntactic incompleteness for S only by strengthening the consistency requirement to ω-consistency (although Rosser later was able to remove this shortcoming). Nonetheless, Gödel's paper did contain the arithmetical semantic incompleteness result for S under the weaker hypothesis of consistency because: (1) he was able to show that the Gödel sentence is unprovable, and (2) he noted that the Gödel sentence is true.[305]

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the second encoding property, *see supra* note 296, that $Prov_S([\pi])$ is a theorem of *EAA*, hence of S. And by the definition of π, we have that $(Prov_S([\pi]) \rightarrow (\neg\pi))$ is a theorem of *EAA* hence of S. Thus, by *modus ponens* we would have that $(\neg\pi)$ is also a theorem of S. This would contradict the consistency of S.

301. Roughly speaking, S is ω-inconsistent if there is some property such that it is provable that there is something satisfying the property, but for anything specifically chosen, S proves that the thing does not satisfy the property. More technically, S is ω-inconsistent if there is a formula $A(x_2)$ such that $(\exists x_2 A(x_2))$ is a theorem of S, but for each natural number p, $(\neg A(\mathrm{p}))$ is a theorem of S. Clearly, ω-consistency implies consistency because an inconsistent system proves all formulas. But consistency does not imply ω-consistency. *See infra* note 309.

Now one establishes the unprovability of the negation as follows. Suppose that $(\neg\pi)$ were a theorem of S. Then by the definition of π, it would follow that $(\exists x_2 Proof_S([\pi],x_2))$ is a theorem of S. Now by the consistency of S it cannot be the case that π is a theorem of S. It follows from (2) of the second encoding property, *see supra* note 296, that for each natural number p, $(\neg Proof_S([\pi],\mathrm{p}))$ is a theorem of *EAA* (hence of S). However, if $(\exists x_2 Proof_S([\pi],x_2))$ is a theorem of S and $(\neg Proof_S([\pi],\mathrm{p}))$ is a theorem of S for all natural numbers p, then S is ω-inconsistent.

302. *See infra* note 309.

303. Rosser changed the formula $P(x_1)$ described *supra* note 295 so that the resulting self-referencing sentence could be read as saying, "If there is a proof of me in the system S, then there is an earlier proof in S of my negation." For the details, see Kleene, *supra* note 172, at 208–09; Mendelson, *supra* note 228, at 144–46.

304. *See supra* note 280 and accompanying text.

305. That is, the Gödel sentence is true as a matter of classical interpretation. Gödel's original paper emphasized syntactic rather than semantic incompleteness, although he did mention semantic incompleteness. *See* Gödel, *supra* note 274, at 596–99. He was hesitant to enter into a debate on the nature of mathematical truth in a climate he believed to be dominated by Formalist ideas. *See* Solomon Feferman, *Kurt Gödel: Conviction and Caution*, in *Gödel's Theorem in Focus* 96, 106–08 (S.G. Shanker ed., 1988) [hereinafter Shanker]. By 1934, he was more explicit. *See* Kurt Gödel, *On Undecidable Propositions of Formal Mathematical Systems*, in *The Undecidable* 41, 64–65 (Martin Davis ed., 1965).

Establishing Gödel's Second Theorem[306] requires in essence a more sophisticated *EAA* with additional encoding features that allow the *EAA* to encode part of the argument for the First Theorem.[307] As stated above, Gödel used the consistency of *S* to show that π is not a theorem of *S*.[308] Using the additional encoding features, it can be shown that *EAA* can follow enough of the argument so that the formal arithmetic counterpart of "If *S* is consistent then π is not provable in *S*" is a theorem of *EAA*. By the definition of π, however, this means that the formal arithmetic counterpart of "If *S* is consistent then π" is a theorem of *EAA*, and hence of *S*. But then *S* cannot prove the formal counterpart of its own consistency else it would prove π, violating the First Theorem.[309]

Metatheoretically, the argument for truth is as follows. *EAA*s are sound. Thus, $(\pi \leftrightarrow (\neg Prov_S([\pi])))$ is true. Now it has just been seen that π is not a theorem of *S*. The truth of $(\neg Prov_S([\pi]))$ follows from a metatheoretical consideration of its construction. Thus, π is true.

Why is this only a metatheoretical argument? As described above, there are three types of systems under consideration: (1) the informal that is to be captured; (2) the formal that is to do the capturing, and (3) the metatheoretical. *See supra* note 254 and accompanying text. What has just been presented is not a system (1) argument for the truth of π, but rather a system (3) argument. However, one can choose to turn such a system (3) argument into an informal system argument by restricting the discussion to (Gödel) numbers. *Cf.* Kleene, *supra* note 172, at 206.

306. Interestingly, Gödel himself sketched but did not prove his Second Theorem. *See* Gödel, *supra* note 274, at 614–16. He intended to provide the details in a later paper, but the paper never did appear. *See id.* at 616 & n.68a.

307. The approach here is based on Smorynski, *supra* note 253.

308. *See supra* note 300 and accompanying text.

309. Specifically, one takes a particular formal counterpart of consistency, call it CON_S. For example, one may take CON_S to be the statement asserting that it is not the case that *S* proves π and *S* proves $(\neg\pi)$, where π denotes the Gödel sentence for *S*. Thus, CON_S would be the formula $(\neg(Prov_S([\pi]) \wedge Prov_S([(\neg\pi)])))$. Given the discussion of consistency *supra* note 238 and accompanying text, there are other choices. They are all provably equivalent in the type of *EAA* envisioned by the Second Theorem. That is, if one takes a CON'_S based on one of the other definitions, then $(CON_S \leftrightarrow CON'_S)$ is a theorem of *EAA*.

The heart of the Second Theorem is showing that $(CON_S \rightarrow \pi)$ is a theorem of *EAA*. Hence $(CON_S \rightarrow \pi)$ is a theorem of *S*. So *S* cannot prove CON_S—else *S* would prove π, violating the First Theorem.

Now how does one show that $(CON_S \rightarrow \pi)$ is a theorem of *EAA*? This gets *really* technical!

As mentioned in the text, one needs an *EAA* with additional properties. With them, *EAA* can encode enough of (1) of the second encoding property described *supra* note 296. Basically, such an *EAA* proves the formal counterpart of, "If *S* proves α, then *S* proves that *S* proves α." Specifically, if α is a formula, then the following is a theorem of such an *EAA*:

$(Prov_S([\alpha]) \rightarrow Prov_S([Prov_S([\alpha])]))$, where $[\alpha]$ is the Gödel number of α.

Also, such an *EAA* can encode *modus ponens*. Basically, such an *EAA* proves the formal counterpart of, "If *S* proves α and *S* proves $(\alpha \rightarrow \beta)$, then *S* proves β." Specifically, if α and β are formulas, then the following is a theorem of such an *EAA*:

$((Prov_S([\alpha]) \wedge Prov_S([(\alpha \rightarrow \beta)])) \rightarrow Prov_S([\beta]))$.

In light of the above discussion, some of the descriptions of Gödel's Theorems that appear in the legal literature are what only can be called confused. One article tells us that "as Kurt Gödel demonstrated, any formal logical system ultimately rests on some undecidable—that is,

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With these two additional encoding properties, one can show that $(CON_S \rightarrow \pi)$ is a theorem of such an *EAA*. In essence, one begins by encoding the argument given *supra* note 300, reproduced as follows with annotations keyed to the discussion below:

> Suppose π were a theorem of *S*. It follows from (1) of the second encoding property that $Prov_S([\pi])$ is a theorem of *EAA*, hence of *S* (*). And by the definition of π, we have that $(Prov_S([\pi]) \rightarrow (\neg\pi))$ is a theorem of *EAA* hence of *S* (**). Thus, by *modus ponens* we would have that $(\neg\pi)$ is also a theorem of *S* (***). This would contradict the consistency of *S* (****).

Then one proceeds as indicated in the text, reproduced as follows with annotations keyed to the discussion below:

> Using the additional encoding features, it can be shown that *EAA* can follow enough of the argument so that the formal counterpart of "If *S* is consistent then π is not provable in *S*" is a theorem of *EAA* (*****). By the definition of π, however, this means that the formal counterpart of "If *S* is consistent then π" is a theorem of *EAA*, hence of *S* (******).

Here we go!

By the third encoding property, the following is a theorem of *EAA*:

(*) $(Prov_S([\pi]) \rightarrow Prov_S([Prov_S([\pi])]))$.

By the definition of π and (1) of the second encoding property, the following is a theorem of *EAA*:

(**) $Prov_S([(Prov_S([\pi]) \rightarrow (\neg\pi))])$.

By the fourth encoding property and the fact that (*) and (**) are theorems of *EAA*, the following is a theorem of *EAA*:

(***) $(Prov_S([\pi]) \rightarrow Prov_S([(\neg\pi)]))$.

Hence the following is a theorem of *EAA*:

$(Prov_S([\pi]) \rightarrow (Prov_S([\pi]) \wedge Prov_S([(\neg\pi)])))$.

Hence by the definition of CON_S the following is a theorem of *EAA*:

(****) $(Prov_S([\pi]) \rightarrow (\neg CON_S))$.

Hence the following is a theorem of *EAA*:

(*****) $(CON_S \rightarrow (\neg Prov_S([\pi])))$.

Hence by the definition of π the following is a theorem of *EAA*:

(******) $(CON_S \rightarrow \pi)$.

Having established the Second Theorem, we finish with an observation and three remarks. Observe that by the definitions of CON_S and π, $(\pi \rightarrow CON_S)$ is a theorem of *EAA*. Using this observation and (******), we remark that *EAA* proves that the Gödel sentence for *S* is equivalent to CON_S. As an additional remark, one can use this observation to see why the Gödel sentence will not suffice to establish syntactic incompleteness. Take a system *T* to which the Second Theorem applies. Consider the system *T'* obtained by adding the axiom $(\neg CON_T)$. The consistency of *T'* follows from the fact that *T* does not prove CON_T. *See* Mendelson, *supra* note 228, at 63. Moreover, it is not difficult to see that *T'* proves its own inconsistency, hence, by the observation, the negation of its Gödel sentence. Finally, we remark that such a *T'* is consistent but, because it proves the negation of its Gödel sentence, ω-inconsistent by Gödel's original version of the First Theorem.

unprovable—propositions."[310] Another refers to "Gödel's proof of ultimate inconsistency in mathematics."[311] A third explains that "A 'complete' theorem is inconsistent if it is an axiom. Nothing purely complete is proved."[312]

Other descriptions have problems with the subtleties of the First Theorem. Some articles have trouble with the definition of an undecidable sentence[313] or the distinction between truth (semantics) and

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310. Loevinger, *supra* note 5, at 343. *See also* Richard A. Givens, *Manual of Federal Practice* § 9.26 (4th ed. Supp. March 1995) (characterizing Gödel's Theorem as showing that "no formal system can describe even in theory all of the information needed for its operation"); Boris I. Bittker, *The Erwin Griswold Lecture*, 11 Am. J. Tax Pol'y 213, 216 (1995) ("Goedel was a turn-of-the-century mathematician who looked at a number of mathematical propositions and proved, at least to the satisfaction of people who understand these things, that certain of those propositions could never be proved as either true or false.").

311. Aoki, *supra* note 6, at 382 (quoting Venturi, *supra* note 6, at 16).

312. Jaffee, *supra* note 7, at 1193. Purcell describes Gödel's results as follows:

> [Gödel] demonstrated to the satisfaction of most of his colleagues that it was theoretically impossible to produce any final or ultimate solution to the problem of the foundations of mathematical logic.

Purcell, *supra* note 51, at 56.

In the quotation above, Purcell cites Parsons but what Parsons says is that "[t]he first theorem ... undermines most attempts at a final solution to the problem of foundations by means of mathematical logic." Parsons, *supra* note 253, at 208. What Parsons means by "mathematical logic" is Hilbert's approach.

Given these two statements and the comment *supra* note 285, the reader might try to evaluate the following:

> [T]here are good reasons to handle the concept of rationality with caution: at least since Goedel proposed his eponymous theorem, there has been good reason to believe that mathematics, supposedly the purest expression of human reason, rests at bottom on begged questions rather than logical proof. At least that is one interpretation of the theorem's implications. [Gödel's First Theorem] deals only with the possibility of establishing the logical foundations of the real number system.

Paul B. Stephan III, *Interdisciplinary Approaches to International Economic Law: Barbarians Inside the Gate: Public Choice Theory and International Economic Law*, 10 Am. U. J. Int'l L. & Pol'y 745, 750–51 & n.5 (1995).

313. Rogers and Molzon seem to imply that a statement that is true but unprovable is undecidable:

> [Gödel] proved that if a number theory system's set of axioms is complex enough to include simple arithmetic, then there are true statements within the system that cannot be reached using the axioms and rules of the system. In other words, he proved that such systems have formally undecidable propositions.

Rogers & Molzon, *supra* note 4, at 993. However, if one takes the system T' described *supra* note 309, the consistency sentence for T' is true, but its negation is provable. In this regard, one also might want to consider the statement that "in any consistent system the statement that the system is consistent ... is ... undecidable." Roy L. Stone-de Montpensier, *Logic and Law: The Precedence of Precedents*, 51 Minn. L. Rev. 655, 662 (1967).

For another example of a confused notion of undecidable, consider the statement that "mathematical undecidables are not easy to find, and there is always the possibility that they will theoremhood (syntax).[314] The most important errors, however, have to do with the limits of what the First Theorem says. For the conclusion that a

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someday be proved, *e.g.*, the four-color mapping problem (recently proved by computer)." D'Amato, *supra* note 284, at 173 n.80.

314. It is unfortunate to see statements such as:

> Godel's Theorem demonstrates that any formalization of arithmetic will be incomplete. That is, no matter what axioms one chooses as the basis from which to prove the truths of arithmetic, there will always exist propositions that can neither be proved true nor false. There will always be gaps, and the addition of further axioms for arithmetic will not fill the gaps. Thus, an infinity of unprovable propositions will always remain.

Kevin W. Saunders, *Realism, Ratiocination, and Rules*, 46 Okla. L. Rev. 219, 219 (1993). For other examples, see Bittker, *supra* note 310, at 216 ("Goedel was a turn-of-the-century mathematician who looked at a number of mathematical propositions and proved, at least to the satisfaction of people who understand these things, that certain of those propositions could never be proved as either true or false."); Lea Brilmayer, *Wobble, or the Death of Error*, 59 S. Cal. L. Rev. 363, 370 n.8 (1986) ("If a system is logically incomplete, then there are statements that are neither provably true nor provably false."); Girardeau A. Spann, *Secret Rights*, 71 Minn. L. Rev. 669, 698 n.58 (1987) [hereinafter Spann, *Secret*] ("Gödel has demonstrated that within closed, consistent logical systems having a threshold level of complexity and sophistication, there exist formally undecidable statements—propositions whose truth or falsity can never be proven."); Girardeau A. Spann, *Deconstructing the Legislative Veto*, 68 Minn. L. Rev. 473, 540 (1984) [hereinafter Spann, *Deconstructing*] (stating that Gödel's work showed mathematicians that "the categories of 'true' and 'false' were not exhaustive"); John Stick, *Can Nihilism Be Pragmatic?*, 100 Harv. L. Rev. 332, 366 n.146 (1986) ("In logic, 'incomplete' means that in any theory that attempts to formalize the area, there will be sentences that cannot be proved either true or false."); Kelso, *supra* note 11, at 831 ("Gödel's theorem is proven by constructing a function from within the set of permissible functions which . . . neither can be included as true nor rejected as false on logical grounds.").

Others waver between relatively precise distinctions and potentially misleading colloquialisms. Such an expository approach is not objectionable when dealing with an audience that has some familiarity with the topic of discussion, but it is inadvisable when dealing with readers who have no prior exposure.

At one point, Brown & Greenberg, for example, do quote Roger Penrose's discussion of the distinction between syntax and semantics. *See* Brown & Greenberg, *supra* note 4, at 1466 n.147.

On the other hand, they make the following statement:

> Consider a classical formal system such as arithmetic. . . . Its proof rules implement certain logical operations, such as "if P and Q are true then P is true," as well as basic applications of the arithmetic operations . . . , such as "$x + y = y + x$." These axioms and arithmetical operations provide the recipe for deducing theorems and truths about the system.

Id. at 1445 (footnotes omitted). They also say that "[a] 'formal system' . . . is a reasoning process designed to . . . deduce truths." *Id.* Finally, they say that "[t]he ability to prove a single inconsistency would so infect the entire system that all propositions (and their negations) would be true!" *Id.* at 1448.

At one point, Rogers & Molzon say that "the concepts of truth and derivation are not at all equivalent." Rogers & Molzon, *supra* note 4, at 996. They also say that "[f]rom a limited number of axioms, the number theorist—like the mathematician generally—develops (proves) other statements (theorems)." *Id.* at 993. On the other hand, they say that "[i]n a particular 'system,' propositions must be expressed in a certain way and a particular set of axioms (assumed truths) and rules is used to generate a set of properly expressed statements that are 'true.'" *Id.* They further say that "[i]t is possible that every consistent *system* of statements that is expressive enough to include self-consistent system S with a language, logical axioms, and rules of inference of the type indicated above is syntactically incomplete, the theorem has two requirements: (1) S must satisfy the ceiling requirement that it is simple enough that the question whether an arbitrary formula is a non-logical axiom can be answered algorithmically and (2) the floor requirement that it is complex enough to contain the formal counterparts of a certain amount of elementary arithmetic. One can see how these requirements flow out of the balancing that is itself a product of the intellectual context of the Theorem; this is one reason why history is important. Not all formal systems satisfy these requirements. For example, the system whose non-logical axioms are the classically true arithmetic sentences is sound, syntactically complete, and semantically complete (hence includes *EAA* and so satisfies the floor requirement). But this set of non-logical axioms is not recursive, and hence the system does not satisfy the ceiling requirement.[315] Moreover, if one restricts the language to addition (or to multiplication), then with respect to this restriction the system whose non-logical axioms are the classically true sentences is sound, syntactically complete, and semantically complete. In addition, this set of non-logical axioms is recursive (hence the system satisfies the ceiling requirement).[316] But this set of non-logical axioms does not include *EAA*, and hence the system does not satisfy the floor requirement. Note also that Gödel's First Theorem does not assert that every consistent formal system is syntactically incomplete. Indeed, the Lindenbaum Lemma contradicts any such assertion.[317] Nonetheless, descriptions in a number of law articles indicate little or no sensitivity to

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referential statements includes some statements that turn on themselves in this way—statements that cannot be proved true or false within the system." *Id.*

For another example, *compare* Dow, *supra* note 4, at 713 & n.29 (separating "formally demonstrable" and "true") *with id.* at 712 ("Gödel proved '. . . that within any consistent formal system, there will be a sentence that can neither be proved true nor proved false . . . .'" (quoting R. Monk, *Ludwig Wittgenstein: The Duty of Genius* 295 n.* (1990))).

315. *See* Hamilton, *supra* note 280, at 154–55.

316. *See* Hao Wang, *From Mathematics to Philosophy* 174 (1974).

317. *See supra* note 287. Whether any system not subject to Gödel's Theorems and their generalizations can serve as a suitable foundational vehicle for mathematics is yet to be determined. *See* Fraenkel et al., *supra* note 3, at 313. In any case, there are those who maintain that "quite a large part" of Hilbert's Program actually survives Gödel's results. *See* Simpson, *supra* note 184, at 353. What they offer are formal systems within which one can carry out quite a bit of infinitary reasoning and for which one can carry out a modified version of the conservation goal. *See* Solomon Feferman, *Hilbert's Program Relativized: Proof-Theoretical and Foundational Reductions*, 53 J. Symbolic Logic 364 (1988); Sieg, *supra* note 272; Simpson, *supra* note 184. For another discussion of some things that can be done with this Article's version of Hilbert's Program in the wake of Gödel, see Prawitz, *supra* note 253, at 262–72. For a different view of Hilbert's Program and its defense in the wake of Gödel, see Detlefsen, *supra* note 253.

the limits of what Gödel's First Theorem says. One article tells us that "to Gödel we owe the insight that every mathematical system contains 'undecidable arithmetic propositions'."[318] Another article says that "in mathematics, a system of explanation cannot be both complete and consistent."[319]

"Grading" descriptions of Gödel's First Theorem is futile, but some generalizations can be made along the following lines. As in the two preceding quotations, the descriptions in some articles more or less miss the boat entirely, recognizing neither a ceiling nor a floor requirement.[320] Some authors arguably allude to one or more aspects of the requirements, but it is not always clear that the authors realize the significance.[321] Many

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318. John A. Scanlan, *Aliens in the Marketplace of Ideas: The Government, the Academy, and the McCarran-Walter Act*, 66 Tex. L. Rev. 1481, 1525 (1988).

319. Harold A. McDougall, *Social Movements, Law, and Implementation: A Clinical Dimension for the New Legal Process*, 75 Cornell L. Rev. 83, 89 n.36 (1989).

320. *See* Banner, *supra* note 45, at 253 n.33 ("[Gödel] proved that a mathematical/logical system cannot be both complete and consistent; that is, a system cannot contain both *all* true propositions and *only* true propositions."); Steven P. Goldberg, *On Legal and Mathematical Reasoning*, 22 Jurimetrics 83, 87 n.26 (1981) ("Gödel's incompleteness theorem . . . establishes that within any axiomatic system there is a statement S such that neither S nor not-S is a theorem."); Susan K. Houser, *Metaethics and the Overlapping Consensus*, 54 Ohio St. L.J. 1139, 1152 (1993) ("Godel proved that a mathematical system is inherently incomplete in that there are assertions that can never be either proved or disproved within the system of known mathematics."); Nancy Levit, *Ethereal Torts*, 61 Geo. Wash. L. Rev. 136, 136 n.3 (1992) ("[A]ny internally consistent mathematical system will be incomplete, in that the system will contain some unprovable propositions."); Rudolph J. Peritz, *Computer Data and Reliability: A Call for Authentication of Business Records Under the Federal Rules of Evidence*, 80 Nw. U. L. Rev. 956, 999 n.214 ("Some sixty years ago, mathematician and logician Kurt Gödel published his Incompleteness Theorem, a demonstration of the necessary incompleteness of sound formal systems . . . ."); Spann, *Deconstructing*, *supra* note 314, at 540 (stating that Gödel's work showed mathematicians that "the categories of 'true' and 'false' were not exhaustive"); Roy Stone, *Affinities and Antinomies in Jurisprudence*, 1964 Cambridge L.J. 266, 281 ("Gödel's theorem . . . says that where in a logical system a statement in the system is provable, it is refutable in the system, and where it is refutable it is provable."); Greenwood, *supra* note 237, at 576 ("In pure mathematics . . . Gödel demonstrated that a system cannot be both consistent and complete . . . .").

321. *See* Stevens v. Tillman, 855 F.2d 394, 399 (7th Cir. 1988) ("Even axiomatic math cannot yield 'factual' (logically true) statements about all interesting arithmetical relations, as Gödel and Turing established."), *cert. denied*, 489 U.S. 1065 (1989); Kornstein, *supra* note 9, at 126 ("Gödel proved that a logical system that has any richness can never be complete . . . ."); Dan L. Burk & Barbara A. Boczar, *Biotechnology and Tort Liability: An Industry at Risk*, 55 U. Pitt. L. Rev. 791, 825 (1994) ("In mathematics . . . the work of Kurt Gödel disclosed the disturbing proposition that no formal system that includes at least arithmetic can be both complete and consistent."); Craig Calhoun, *Social Theory and the Law: Systems Theory, Normative Justification, and Postmodernism*, 83 Nw. U. L. Rev. 398, 408 (1989) (referring to "Kurt Godel's proof of the insufficiency of the arithmetic postulates"); Vivian G. Curran, *Deconstruction, Structuralism, Antisemitism and the Law*, 36 B.C. L. Rev. 1, 28 n.71 (1994) (stating that Gödel demonstrated "'the impossibility of constructing a theoretical system within which all true statements of number theory are theorems'" (quoting Jonathan Culler, *On Deconstruction: Theory and Criticism After Structuralism* 133

(1982))); Anthony D'Amato, *Can Legislatures Constrain Judicial Interpretation of Statutes?*, 75 Va. L. Rev. 561, 597 (1989) ("Gödel . . . proved that there are some mathematical propositions . . . that can neither be proved nor disproved within a mathematical system of at least enough complexity as to include ordinary arithmetic."); D.H. Kaye, *The Logic and Anti-Logic of Secret Rights*, 72 Minn. L. Rev. 603, 617 n.71 ("Gödel's proof does demonstrate that there are arithmetic truths that cannot be proved within a strictly formal system."); Warren Lehman, *Rules in Law*, 72 Geo. L.J. 1571, 1592 & n.59 (1984) ("[E]ven arithmetic cannot be treated as a closed system. . . . Goedel's Theorem . . . announced the impossibility in mathematics of a completely closed system."); Jeanne L. Schroeder, *Subject: Object*, 47 U. Miami L. Rev. 1, 53 (1992) ("Goedel's Incompleteness Theorem . . . mathematically proves that all statements within a fixed number system cannot be mathematically proved."); Spann, *Secret*, *supra* note 314, at 698 n.58 ("Gödel has demonstrated that within closed, consistent logical systems having a threshold level of complexity and sophistication, there exist formally undecidable statements—propositions whose truth or falsity can never be proven."); Stick, *supra* note 314, at 366 n.146 ("Kurt Gödel demonstrated by means of his incompleteness theorems that even elementary areas of mathematics such as first-order arithmetic are incomplete."); John T. Valauri, *The Concept of Neutrality in Establishment Clause Doctrine*, 48 U. Pitt. L. Rev. 83, 124 n.215 (1986) ("Even in arithmetic, Gödel has shown, one cannot have both completeness and consistency."); Williams, *supra* note 12, at 439 (stating that Gödel's "incompleteness theorem" demonstrates "that arithmetic cannot be both complete and internally consistent"); Jennifer L. Orff, Note, *Demanding Justice Without Truth: The Difficulty of Post Modern Feminist Legal Theory*, 28 Loyola L.A. L. Rev. 1197, 1202 (1995) (quoting Williams, *supra* note 12, at 439); Veilleux, *supra* note 81, at 1997 ("Gödel's Theorem demonstrates the impossibility of a complete, consistent logical system of any complexity."); Mary I. Coombs, *Lowering One's Cites: A (Sort of) Review of the University of Chicago Manual of Legal Citation*, 76 Va. L. Rev. 1099, 1102 n.16 (1990) (book review) ("[C]omplex mathematical systems can never be both self-contained and complete."); Bernard E. Jacob, *Ancient Rhetoric, Modern Legal Thought, and Politics: A Review Essay on the Translation of Viehweg's "Topics and Law"*, 89 Nw. U. L. Rev. 1622, 1657 n.126 (1995) (book review) ("Gödel's Proof is a widely accepted metamathematical theorem that shows that no mathematical axiomatic system of complexity is 'complete'."); David E.B. Smith, *Just When You Thought It Was Safe to Go Back into the Bluebook: Notes on the Fifteenth Edition*, 67 Chi.-Kent L. Rev. 275, 275 n.1 (1991) (book review) ("[N]o sufficiently complex mathematical system can simultaneously achieve completeness and consistency.").

It is often difficult to decide whether a quotation should be placed in this or the preceding footnote. For example, it is possible to find some allusion to the floor requirement in Jean W. Burns, *Standing and Mootness in Class Actions: A Search for Consistency*, 22 U.C. Davis L. Rev. 1239, 1287 n.218 ("As Kurt Gödel proved in 1931, mathematics cannot be both internally consistent and complete at the same time . . . ."); Anthony D'Amato, *Legal Uncertainty*, 71 Cal. L. Rev. 1, 44 n.96 (1983) ("[T]he closer that economics might come to mathematical precision, the more likely it is that a 'Godel' problem will arise, where no improvement in the system's postulates will suffice to answer as a formal matter all the economic-law problems that may confront it."); Thomas C. Heller, *Structuralism and Critique*, 36 Stan. L. Rev. 127, 154 n.52 (describing "Goedel's proposition that logic, mathematics, and other complex systems can never be fully described by a closed set of rules"); Christopher D. Stone, *Should Trees Have Standing Revisited: How Far Will Law and Morals Reach? A Pluralist Perspective*, 59 S. Cal. L. Rev. 1, 74 (1985) ("Gödel and others have laid to rest any hope of discovering the one grand and complete set of axioms from which all true statements of mathematics can be derived.").

of those who make some attempt to describe the limits of what Gödel has to say slip into misleading generalities.[322] Very few authors avoid these problems.[323]

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322. Such an expository approach is not objectionable when dealing with an audience that has some familiarity with the topic of discussion, but it is inadvisable when dealing with readers who have no prior exposure. At one point, for example, Saunders does say:

> [Gödel's First Theorem] works in mathematics, because Godel managed to express the metalanguage for arithmetic, the language used to talk about arithmetic, within arithmetic. In order for Godel's Theorem to apply to law, it would seem that the same feat must be accomplished for law. More is required than showing that laws may be self-referential or that law may have rules and metarules. Required is a demonstration that the metalanguage of law—legal English—can in some sense be embedded in the law.

Saunders, *supra* note 314, at 220 (footnotes omitted). On the other hand, he says:

> Godel's Theorem demonstrates that any formalization of arithmetic will be incomplete. That is, no matter what axioms one chooses as the basis from which to prove the truths of arithmetic, there always will exist propositions that can neither be proved true nor false. There will always be gaps, and the addition of further axioms for arithmetic will not fill the gaps. Thus, an infinity of unprovable propositions will always remain.

Id. at 219. *Compare* Dow, *supra* note 4, at 713 (quoting Roger Penrose's precise description of Gödel's First Theorem) *with id.* at 712 ("Gödel proved '. . . that within any consistent formal system, there will be a sentence that can neither be proved true nor proved false . . . .'" (quoting Monk, *supra* note 314, at 295 n.*)); *compare* Kelso, *supra* note 11, at 831–32 (attempting to describe two encoding properties used in establishing Gödel's First Theorem) *with id.* at 834 n.48 ("Gödel's theorem does prevent any mathematical system of axioms from even being complete and consistent."); *compare* Rogers & Molzon, *supra* note 4, at 996 (discussing inclusion of arithmetic and use of encoding properties) *with id.* at 993 ("[I]f a number theory system's set of axioms is complex enough to include simple arithmetic, then there are true statements within the system that cannot be reached using the axioms and rules of the system.") *and id.* at 996 n.9 ("Gödel's incompleteness result holds in any . . . language which is expressive enough to describe arithmetic with addition and multiplication.") *and id.* at 1022 ("Analogy to Gödel's Theorem teaches a more fundamental lesson: any sufficiently expressive formal system must have undecidable propositions."); *compare* John M. Farago, *Intractable Cases: The Role of Uncertainty in the Concept of Law*, 55 N.Y.U. L. Rev. 195, 224–25 (1980) (providing somewhat vague discussion of limitations) *with id.* at 225 ("Gödel demonstrated that a deductive logical system sophisticated enough to express arithmetic is necessarily either inconsistent or incomplete."); *compare* Stone-de Montpensier, *supra* note 313, at 664, 670 n.54 (quoting Hao Wang's description of Gödel's First Theorem and describing importance of recursivity requirement) *with id.* at 662 ("The Gödel result shows that if a statement in a mathematical system is provable it is refutable, and if it is not provable it is not refutable.").

323. Perhaps the best description, so far as it goes, is contained in Brown & Greenberg, *supra* note 4. They quote scientist Roger Penrose's definition of formal system to include a finite set of axioms, *id.* at 1445, and state that "Gödel demonstrated that formal systems powerful enough to express the axioms and propositions of arithmetic cannot be both complete and consistent," *id.* at 1466. Penrose seems to mean a finite set of axiom schemas, but it is clear that his notion is meant to imply that the set of non-logical axioms is recursive. This seems to be the implication that Brown & Greenberg draw as well. *Id.* at 1445-46. They also point out that his argument was "closely tied to the specific formal system he considered." *Id.* at 1467. They also emphasize the existence of limitations. *Id.* at 1467 n.150. For another example, see Brilmayer, *supra* note 314, at 370 n.8 ("The mathematician Gödel proved that certain types of mathematical systems are incomplete . . . .").

There are similar problems with the treatment of the Second Theorem. Once again, the limitations are closely tied to the history. Gödel's Second Theorem does not say that it is impossible to argue for consistency. Metatheoretical arguments for consistency are available for a wide variety of formal systems.[324] Moreover, for certain arithmetic systems there are particular statements of consistency (although not of the type Hilbert envisioned) that are metatheoretically equivalent to the statement used in the Second Theorem,[325] and whose formal counterparts can be proved within the system.[326] Indeed, mathematicians warn that "care needs to be taken in stating exactly what has been proven in this area."[327] Many articles, however, indicate little or no sensitivity to these distinctions. One article tells us, for example, that "Gödel proved that a logical system that has any richness can never be . . . guaranteed to be consistent."[328] Another article says that "[e]ven arithmetic, Kurt Gödel has shown, cannot be shown to be internally logically consistent."[329]

"Grading" descriptions of Gödel's Second Theorem also is futile, but some generalizations can be made along the following lines. Few mention that there exist arguments for the consistency of various

324. Indeed, several years after Gödel's work was published, Gerhard Gentzen provided arguments (although not quite as simple as Hilbert would have liked) for an important arithmetic formal system known as Peano Arithmetic. Gödel's Second Theorem indicates the impossibility of establishing the consistency of Peano Arithmetic by means as simple as Hilbert would have liked (finitary means). What Gentzen did was add to Hilbert's finitistic machinery a certain amount of infinitary machinery. He showed that this new metatheoretical machinery was strong enough to prove the consistency of Peano Arithmetic. *See* Fraenkel et al., *supra* note 3, at 314; Gaisi Takeuti, *Proof Theory* 114 (1987). Gentzen's techniques are what this Article calls proof-theoretic or metamathematical techniques. *See supra* note 254. Moving in the other direction, one can prove the consistency of "large chunks" of Peano Arithmetic by finitary proof-theoretic means. *See* Richard Kaye, *Models of Peano Arithmetic* 140 (1991).

The consistency of the systems described *supra* text accompanying notes 315–16 follows from the fact that a set of sentences true in some model is consistent. For a discussion of this fact, see Jane Bridge, *Beginning Model Theory: The Completeness Theorem and Some Consequences* 67 (1977). Such an argument is what this Article calls a model-theoretic argument. *See supra* note 254.

325. They are not equivalent by means formalizable in the formal system, else the Second Theorem would be violated.

326. *See* Michael D. Resnik, *On the Philosophical Significance of Consistency Proofs*, in Shanker, *supra* note 305, at 115, 123; *see also* Mendelson, *supra* note 228, at 148–49; *Thirty Years of Foundational Studies*, in 1 Andrzej Mostowski, *Foundational Studies: Selected Works* 1, 19–22 (1979).

327. Hunter, *supra* note 233, at 257.

328. Kornstein, *supra* note 9, at 126-27.

329. John T. Valauri, *Confused Notions and Constitutional Theory*, 12 N. Ky. L. Rev. 567, 572 n.25 (1985).

systems.[330] No one mentions the fact that the Second Theorem depends on a particular formulation of consistency.[331] Some authors do not even clearly distinguish between the First and Second Theorems.[332]

Gödel himself was aware of at least some of these problems. He wanted, for example, to use special language to indicate that there were limitations on the applicability of his results. He suggested that the phrase "formal system" only be used to indicate those systems simple enough to satisfy the ceiling requirement.[333] Other mathematical authors use the phrase "axiomatized system" in a restrictive way.[334] In the legal literature, these mathematical terms of art often appear without any

330. For an example of an author who does mention this, see Stone-de Montpensier, *supra* note 313, at 670 n.55.

331. In addition to those descriptions already mentioned, see Curran, *supra* note 321, at 28 n.71 ("'[N]o axiomatic system can even be proved to be fully coherent and consistent from within its own rules and postulates.'" (quoting George Steiner, *Real Presences* 125 (1989))); D'Amato, *supra* note 321, at 597 n.94 (quoting Nagel & Newman, *supra* note 274, at 6 for proposition that Gödel "'proved that is impossible to establish the internal logical consistency of a very large class of deductive systems—elementary arithmetic, for example—unless one adopts principles of reasoning so complex that their internal consistency is as open to doubt as that of the systems themselves'"); Dow, *supra* note 4, at 712 (characterizing Gödel's Second Theorem as showing that "'the consistency of a formal system of arithmetic cannot be proved *within* that system'" (quoting Monk, *supra* note 314, at 295 n.*)); *id.* at 713 (stating that "the consistency of a formal system of arithmetic cannot be proved by any means that is formalizable within that system"); Houser, *supra* note 320, at 1151–52 ("Kurt Godel . . . showed that a mathematical system could only be consistent (non-paradoxical) when viewed from 'outside' the system. That is, a definition of a system must be made independently of the system itself to avoid the paradoxes of self-reference."); Scanlan, *supra* note 318, at 1525 ("[T]o Gödel we owe the insight that every mathematical system . . . is . . . incapable of demonstrating its own 'consistency.'"); Stone-de Montpensier, *supra* note 313, at 664 (describing "impossibility of formalizing any consistency proof" in "any sufficiently rich formal system"); Lawrence H. Tribe, *Taking Text and Structure Seriously: Reflections on Free-Form Method in Constitutional Interpretation*, 108 Harv. L. Rev. 1223, 1291 n.225 (1995) ("The mathematician Kurt Gödel showed that no finite, consistent axiomatic system can provide for the proof of its own consistency." Tribe cites Hofstadter, *supra* note 11, and that source indicates that the use of the word "finite" means that the system encompasses what Hilbert called finitistic methods. *See supra* text accompanying note 269.).

332. For examples, see Brown & Greenberg, *supra* note 4, at 1470 n.160 (quoting comments on Second Theorem by Nagel & Newman, *supra* note 274, at 98 n.31, in midst of discussion of First Theorem); Williams, *supra* note 12, at 439 & n.63 (describing Gödel's First Theorem as demonstrating "that arithmetic cannot be both complete and internally consistent" (citing to discussion of both First and Second Theorems by Nagel & Newman, *supra* note 274, at 85–97), and then stating that "Gödel's analysis did not rule out a metamathematical proof of the consistency of arithmetic . . ." (citing to discussion of Second Theorem by Nagel & Newman, *supra* note 274, at 96–97)).

333. *See* Gödel, *supra* note 274, at 616 n.70.

334. *See, e.g.*, Herbert B. Enderton, *A Mathematical Introduction to Logic* 146 (1972).

explanation of their possible significance.[335] Indeed, few authors seem aware of any such significance.[336]

With respect to the techniques described here for establishing Gödel's Theorems, two points are in order. As mentioned above, the Gödel sentence will not suffice for the syntactic incompleteness part of the First Theorem—the Gödel sentence need not be undecidable.[337] However, the presentations of a number of commentators suggest that the Gödel sentence does suffice.[338] Second, it must be noted that the encoding properties utilized in the arguments described here are limited in scope. Vague statements implying that there are simple ways of encoding every

335. *See* Stevens v. Tillman, 855 F.2d 394, 399 (7th Cir. 1988) ("Even axiomatic math cannot yield 'factual' (logically true) statements about all interesting arithmetical relations, as Gödel and Turing established."), *cert. denied*, 489 U.S. 1065 (1989); Givens, *supra* note 310, at § 9.26 (characterizing Gödel's Theorem as showing that "no formal system can describe even in theory all of the information needed for its operations"); Burk & Boczar, *supra* note 321, at 825 ("In mathematics . . . the work of Kurt Gödel disclosed the disturbing proposition that no formal system that includes at least arithmetic can be both complete and consistent."); Curran, *supra* note 321, at 28 n.71 ("'[N]o axiomatic system can even be proved to be fully coherent and consistent from within its own rules and postulates.'" (quoting Steiner, *supra* note 331, at 125)); Dow, *supra* note 4, at 712 ("Gödel proved '. . . that within any consistent formal system, there will be a sentence that can neither be proved true nor false . . . .'" (quoting Monk, *supra* note 314, at 295 n.*)); Goldberg, *supra* note 320, at 87 n.26 ("Gödel's incompleteness theorem . . . establishes that within any axiomatic system there is a statement S such that neither S nor not-S is a theorem."); Kaye, *supra* note 321, at 617 n.71 ("Gödel's proof does demonstrate that there are arithmetic truths that cannot be proved within a strictly formal system."); Peritz, *supra* note 320, at 999 n.214 ("Some sixty years ago, mathematician and logician Kurt Gödel published his Incompleteness Theorem, a demonstration of the necessary incompleteness of sound formal systems . . . ."); Pierre Schlag, *Fish v. Zapp: The Case of the Relatively Autonomous Self*, 76 Geo. L.J. 37, 40 n.16 (1987) ("'[A]ll consistent axiomatic formulations of number theory include undecidable propositions.'" (quoting Hofstadter, *supra* note 11, at 17)); Schroeder, *supra* note 321, at 53 n.141 ("'[A]ll consistent axiomatic formulations of number theory include undecidable propositions.'" (quoting what she calls the "colloquial" description in Hofstadter, *supra* note 11, at 17)); Tribe, *supra* note 331, at 1291 n.225 ("The mathematician Kurt Gödel showed that no finite, consistent axiomatic system can provide for the proof of its own consistency."); Mark G. Yudof, *In Search of a Free Speech Principle*, 82 Mich. L. Rev. 680, 690 n.33 (1984) (quoting Hofstadter, *supra* note 11, at 24 for the proposition that "no axiomatic system whatsoever could produce all number-theoretic truths, unless it were an inconsistent system"); Jacob, *supra* note 321, at 1657 n.126 ("Gödel's Proof is a widely accepted metamathematical theorem that shows that no mathematical axiomatic system of complexity is 'complete'."); Kelso, *supra* note 11, at 832 n.39 ("Gödel's proof pertains to any non-trivial axiomatic system.").

336. Brown & Greenberg do attach a particular significance to the use of the phrase "formal system." Brown & Greenberg, *supra* note 4, at 1445–46. Stone-de Montpensier perhaps does so as well. *See* Stone-de Montpensier, *supra* note 313, at 670 n.54.

337. *See supra* text accompanying note 302; *supra* note 309.

338. *See* Brown & Greenberg, *supra* note 4, at 1468; Dow, *supra* note 4, at 713; Rogers & Molzon, *supra* note 4, at 993, 996; Kelso, *supra* note 11, at 833 n.44.

John Farago uses the Gödel sentence, but his version of the First Incompleteness Theorem only covers semantic incompleteness. *See* Farago, *supra* note 322, at 225.

metatheoretical concept are misleading and must be highly refined.[339] Indeed, the realization of a difference in the encodability of concepts involving proof and truth helped lead Gödel to his results in the first place.

It is worth examining this second point in some detail. Gödel did not set out to destroy the Hilbert Program. Indeed, his work began with attempts to implement it.[340] Gödel had hoped to use some notion of encoding as the basis for various finitistic consistency arguments.[341] His use of encoding led him to consider what might be done with the following Liar-type Paradox statement: "This statement is false." Gödel realized that definite sense could be given to the phrase "this statement" through self-referencing. He saw that if truth were encodable in a certain manner, he could find a precise version of the Liar statement, giving a

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339. Farago, for example, says the following:

> Gödel demonstrated that even a logical system as simple as arithmetic can express *within* itself a system of analysis *about* itself. He developed a formal mapping of meta-arithmetic onto arithmetic. . . .
>
> The technique Gödel adopted was roughly the following. He initially developed a way in which to associate meta-arithmetical expressions with the numbers of arithmetic; that is, he devised a way to assign numbers (integers) to statements *about* arithmetic.

Farago, *supra* note 322, at 224–25 (footnotes omitted).

Rogers & Molzon say the following:

> One of Gödel's accomplishments was demonstrating that a formal language, arithmetic, could serve as its own metalanguage. He did this by describing a correspondence between statements in arithmetic and statements in the metalanguage. In fact, Gödel described this relationship so precisely that one can think of it as a machine that, when given a metamathematical statement as input, churns out [an arithmetic] statement. The machine can also run in reverse, so that if one plugs in an [arithmetic] statement, out pops the corresponding metamathematical statement.

Rogers & Molzon, *supra* note 4, at 996.

Saunders says the following:

> [Gödel's First Theorem] works in mathematics, because Godel managed to express the metalanguage for arithmetic, the language used to talk about arithmetic, within arithmetic. In order for Godel's Theorem to apply to law, it would seem that the same feat must be accomplished for law. More is required than showing that laws may be self-referential or that law may have rules and metarules. Required is a demonstration that the metalanguage of law—legal English—can in some sense be embedded in the law.

Saunders, *supra* note 314, at 220 (footnotes omitted).

For another example, see Brown & Greenberg, *supra* note 4, at 1469 ("[Gödel] demonstrated that any proposition—or more accurately, any metaproposition about propositions of arithmetic—can be expressed as a statement about numbers, and hence as a statement *within arithmetic*.").

340. *See* Karl Sigmund, *A Philosopher's Mathematician: Hans Hahn and the Vienna Circle*, Mathematical Intelligencer, Fall 1995, at 16, 22–23.

341. *See* Feferman, *supra* note 305, at 105; Hao Wang, *Some Facts About Kurt Gödel*, 46 J. Symbolic Logic 653, 654 (1981).

contradiction. It follows that truth could not be so encoded. However, he realized that certain concepts involving proof were so encodable. Self-referencing applied to these concepts leads to his First Theorem. As has been seen, the main work in the proof of Gödel's First Theorem consists of making these realizations more concrete.[342] Thus, somewhat ironically, Gödel was led to his results through attempts to implement the second step in Hilbert's Program![343]

The reader also should pause and consider the context of Gödel's work—a context that here has been briefly sketched and much simplified. It is worth comparing this presentation to the sparse and sometimes inaccurate accounts presented in the legal literature.[344] Indeed,

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342. Feferman, *supra* note 305, at 105–06. *See also* John W. Dawson, Jr., *The Reception of Gödel's Incompleteness Theorems*, in Shanker, *supra* note 305, at 74, 92 n.5; Gödel, *supra* note 305, at 63–65. In this regard, and recalling the discussion *supra* note 168, one might want to consider the following:

> Russell and Whitehead believed that they could banish paradoxes from mathematics by segregating the component parts of the paradox on different levels of analysis. Gödel's Theorem convinced mathematicians of the impossibility of getting rid of this pattern of circularity, recursive definition, and self-swallowing analysis.

James Boyle, *A Theory of Law and Information: Copyright, Spleens, Blackmail, and Insider Trading*, 80 Cal. L. Rev. 1413, 1444 (1992).

The distinction in the encodability of semantic and syntactic concepts, although clearly anticipated by Gödel, was systematically investigated for the first time in Alfred Tarski, *The Concept of Truth in Formalized Languages*, in *Logic, Semantics, Metamathematics: Papers from 1923–1938* (J.H. Woodger trans., 1956). For more on how one might measure how difficult it is to encode or define a concept, see Peter G. Hinman, *Recursion-Theoretic Hierarchies* (1978).

The results of Gödel, Tarski, and Church, described *supra* note 297, are often referred to as "limitative results." *See* Fraenkel et al., *supra* note 3, at 310–20. Either Church's results or Tarski's results can be used to establish Gödel's First Theorem. *See* Enderton, *supra* note 334, at 227–29 (using Tarski); Shoenfield, *supra* note 226, at 131–32 (using Church). The fact that Church can be used is recognized, somewhat obliquely, in Farago, *supra* note 322, at 228.

343. Hao Wang calls this an example of "problem transmutation." Wang, *supra* note 247, at 56.

344. Jaffee says the following:

> Some positivists (and some "realists") insist that a normative system can obtain from a mere assumption—an hypothesis that the system exists. If we assume the system as if it were a fact, they say, we can believe it without reference to any norm or premise beyond it.
>
> Well, Russell and Whitehead tried to do the same in *Principia Mathematica*, and Gödel's famous proof crumbled the effort.

Jaffee, *supra* note 7, at 1193.

Dow says the following:

> At the turn of the century, some thirty years before Gödel developed these theorems, the mathematician David Hilbert had asked whether it could be proved that the axioms of arithmetic are consistent—that is, whether the finite number of logical steps based upon these axioms could ever lead to contradictory results. Some ten years after Hilbert posed the question, Bertrand Russell and Alfred Whitehead published the first volume of *Principia Mathematica*. This work endeavored to prove that all pure mathematics can be derived from a small number of

the technical mischaracterizations of Gödel's work illustrate what can occur when mathematical results are torn from their intellectual heritage.

6. *Final Comments*

This section illustrates what is entailed in overcoming the first hurdle in doing meaningful interdisciplinary work—namely, gaining an understanding of what is often a foreign discipline. Such an understanding also helps address the second hurdle because a detailed study of a part of another discipline often reveals that discipline's relevance and separateness. Those who doubt the necessity of such an undertaking might want to consider the next section.

C. *Legal Implications*

1. *The Current Foundational Crisis in Mathematics and the Critique of Legal Science*

Legal scholars have invoked the foundational crisis in mathematics in their own specific foundational debates. In particular, they have applied Gödel's work to law and have drawn a variety of specific conclusions about legal reasoning. For some, the applications help establish or highlight the "indeterminacy" or "uncertainty" present in legal analysis

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fundamental logical principles. The *Principia* failed in this quest; it also did not answer Hilbert's question; Gödel's Theorem addressed the system described in the *Principia*.

Dow, *supra* note 4, at 713–14.

For other examples, see Kornstein, *supra* note 9, at 126 ("[T]here had been many attempts by mathematicians and philosophers to mechanize the thought processes of reasoning, always stressing the completeness and consistency."); Banner, *supra* note 45, at 253 n.33 ("While the Langdellians were attempting to reformulate law as an organized group of propositions flowing from a small number of fundamental axioms, the exact same thing was occurring in mathematics, an effort which culminated in A.N. Whitehead & Bertrand Russell, *Principia Mathematica* . . . . Gödel proved the impossibility of the project in 1931 . . . ."); Farago, *supra* note 322, at 224 ("[Gödel's work] relates to a series of paradoxes within logical thought that became increasingly troublesome as logic became increasingly formalized."); Saunders, *supra* note 314, at 229 ("[Gödel's First Theorem] dealt a blow to the Hilbert Program of developing a formalization that would capture all of mathematics.").

Brown & Greenberg focus on Hilbert's interest in the *Entscheidungsproblem* described *supra* note 247 and accompanying text. *See* Brown & Greenberg, *supra* note 4, at 1466. Now it is the case that with hindsight we can see that the unsolvability of the *Entscheidungsproblem* follows from work contained in Gödel's 1931 paper and a result in a 1935 paper of Kleene. *See* Davis, *supra* note 305, at 109. But this was not noticed at the time, and as described *supra* note 297, the work of Turing and Church are credited with establishing the unsolvability of the *Entscheidungsproblem*. Gödel's 1931 paper was focused on Hilbert's Program, not the *Entscheidungsproblem*. This distinction in intellectual history is recognized in Roger Penrose, *The Emperor's New Mind* 34 (1989), the very source cited by Brown & Greenberg for their historical remarks.

(what is here called legal science).[345] For others, the applications show no more than that criticisms based on indeterminacy are "unfair."[346] For still others, a close examination of Gödel's work actually shows how to overcome any such problems.[347] The intent here is not to evaluate the ultimate conclusions, but merely to examine the machinery used to obtain them. In particular, no attempt is made to evaluate the various notions of indeterminacy and uncertainty that have appeared in the general legal literature.[348]

Scholars "apply" Gödel's work in several ways. Some claim that the hypotheses of something like his First Theorem are satisfied in the legal setting, while others create legal analogues of the Gödel sentence or analyze the argument establishing the First Theorem.

Satisfying the hypotheses of such a theorem in a legal context would require several steps. One would have to provide a formal system for law. In particular, one would have to provide a legal language, which in turn requires specifying an alphabet and the set of legal formulas; a set of legal axioms; and a set of rules of legal inference. One need only examine the formal system provided for propositional logic to get an idea of the difficulty that this would entail.[349] This, however, is not sufficient in and of itself. The resulting legal formal system must be properly

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345. *See* Kornstein, *supra* note 9, at 127; Colin H. Buckley, *Issue Preclusion and Issues of Law: A Doctrinal Framework Based on Rules of Recognition, Jurisdiction and Legal History*, 24 Hous. L. Rev. 875, 904 n.166 (1987); D'Amato, *supra* note 321, at 597–603; Veilleux, *supra* note 81, at 1997. *Cf.* Brown & Greenberg, *supra* note 4, at 1487-88; Farago, *supra* note 322, at 236–39; Goldberg, *supra* note 320, at 90.

346. *See* Rogers & Molzon, *supra* note 4, at 992, 1022.

347. *See* Stone-de Montpensier, *supra* note 313, at 670.

348. For some taxonomies of indeterminacy with varying meanings, see Jules L. Coleman & Brian Leiter, *Determinacy, Objectivity, and Authority*, 142 U. Pa. L. Rev. 549, 559–78 (1993) (discussing indeterminacy of reasons and indeterminacy of causes); Dow, *supra* note 4, at 716 (discussing linguistic indeterminacy, formal indeterminacy, and conceptual indeterminacy); Ken Kress, *A Preface to Epistemological Indeterminacy*, 85 Nw. U. L. Rev. 134 (1990) [hereinafter Kress, *Preface*] (discussing epistemological indeterminacy and metaphysical indeterminacy); Ken Kress, *Legal Indeterminacy*, 77 Cal. L. Rev. 283 (1989) (discussing radical indeterminacy and moderate indeterminacy); John A. Miller, *Indeterminacy, Complexity, and Fairness: Justifying Rule Simplification in the Law of Taxation*, 68 Wash. L. Rev. 1, 10–11 (1993) (discussing practical indeterminacy and theoretical indeterminacy); Christopher L. Kutz, Note, *Just Disagreement: Indeterminacy and Rationality in the Rule of Law*, 103 Yale L.J. 997, 999–1002 (1994) (discussing underdeterminacy and rational indeterminacy). For some general comments on indeterminacy, see Miller, *supra*, at 10–11 & nn.33–36.

For discussions of uncertainty, see D'Amato, *supra* note 320, at 2 (discussing legal uncertainty); John M. Farago, *Judicial Cybernetics: The Effects of Self-Reference in Dworkin's Rights Thesis*, 14 Val. L. Rev. 371, 385 (1980) (discussing "four sources of . . . uncertainty").

349. *See supra* note 228.

balanced to assure something like the encoding properties previously described.[350] Moreover, if one wants to invoke semantic incompleteness, then a notion of truth will have to be developed as well. Finally, recall that the precise conclusion given by the proof of Gödel's First Theorem is the existence of an undecidable *arithmetic* sentence.[351] Presumably, a more legal-type conclusion would have to follow for an undecidability result to be of particular interest in legal foundational debates. What of efforts to do all this? About the best that can be said is that "[t]hough some theorists suppose this . . . is feasible, no effort . . . has come even remotely close to accomplishing this feat."[352] Ignoring these types of problems can result in making the mistake of turning a "law and [discipline X]" study into a "law as (a subset of) [discipline X]" study. One must consider carefully statements such as, "The implications of Gödel's Theorem for any theory of law have been ignored for too long . . . . Every theory of law is incomplete."[353]

350. *See supra* notes 295–97 and accompanying text.

351. *See supra* text accompanying note 284.

352. Dow, *supra* note 4, at 715. *See* Banner, *supra* note 45, at 244 n.4; Kaye, *supra* note 321, at 617 n.71; Saunders, *supra* note 314, at 220; M.B.W. Sinclair, *Notes Toward a Formal Model of Common Law*, 62 Ind. L.J. 355, 363 n.33 (1987); Spann, *Secret*, *supra* note 314, at 698 n.58; Stick, *supra* note 314, at 366 n.146; Kelso, *supra* note 11, at 834 n.48; *see also* Brown & Greenberg, *supra* note 4, at 1462, 1472–74 (considering project to be feasible but difficult).

353. Kornstein, *supra* note 9, at 127. Kornstein provides no details. He is cited with approval in Veilleux, *supra* note 81, at 1997 nn.132–33.

Goldberg, although noting that judicial reasoning is often only a "parody of a mathematical theorem," Goldberg, *supra* note 320, at 86, goes on to assert without elaboration that judges use "the axiomatic method" which is subject to "Gödel's incompleteness theorem . . . that emphasize[s] the limitations of what axiom systems can do," *id.* at 90.

D'Amato opines that "[a]ny existing language qualifies as a system of at least as much complexity as ordinary arithmetic, and hence Gödel's proof applies to legal, textual, and linguistic demonstrations." D'Amato, *supra* note 321, at 597. This is not much more help. Moreover, he cites Raymond Smullyan as support for this somewhat vague generalization. I have read the indicated citation, and I am unable to see any support there; Smullyan is quite precise about the types of systems to which his versions of Gödel's Theorems apply. Levit similarly cites Smullyan without explanation. Levit, *supra* note 320, at 136 n.3.

Farago suggests that "some jurisprudential logician could use Gödel's Proof as a paradigm to do to law what Gödel did to mathematics, i.e., demonstrate the necessary incompleteness or inconsistency of the legal systems." Farago, *supra* note 322, at 228–29. However, he does add something by alluding to the encoding features in the argument establishing Gödel's First Theorem: "self-reference is not just a part of the law; it is essential to it," and "like arithmetic, it must be that to the extent we can capture 'law' within a formal system, we also will be able to express 'meta-law' within that same system." *Id.* at 226. But he admittedly does not carry out the details. *Id.* at 226, 229 n.141.

Similarly, Rogers & Molzon assert that "Gödel's theorem at least suggests (and by analogy proves) that all systems of law permit the construction of undecidable propositions." Rogers & Molzon, *supra* note 4, at 1014. Indeed, they cite Farago and Kornstein as examples of earlier efforts. *Id.* at 997 n.11. Rogers & Molzon go on to refer to the encoding features saying that "[i]t is in fact

Some scholars hope to avoid these problems by producing a specific incompleteness based on legal versions of the Gödel sentence itself. Two scholars offer the following legal version of the Gödel sentence:

> "[Y]ields a statement for which, when presented to a court as an example of an indeterminate proposition of the law under circumstances where the question of its determinacy is necessary to resolve the dispute in question, the law does not compel a court to determine that it is true, when appended to its own quotation" yields a statement for which, when presented to a court as an example of an indeterminate proposition of the law under circumstances where the question of its determinacy is necessary to resolve the dispute in question, the law does not compel a court to determine that it is true, when appended to its own quotation.[354]

Other examples are culled from extant legal problems, such as whether certain highest court decisions can bind the highest court.[355] As a preliminary matter, it is not clear that these scholars have avoided all of the criticisms described above. Defining words such as "statement," "indeterminate," "proposition," and "true," would lead to the difficulties

easy to conclude that legal rules can be self-referential" and that "statements about the law can be in the form of laws." *Id.* at 1010. But outside of isolated examples, the details are lacking.

D'Amato offers an interesting twist by arguing in a later piece that "[a]lthough [it] is technically correct in saying that [Gödel's First Theorem was] designed to apply to formal systems, my position is that either [Gödel's First Theorem applies] a fortiori to non-formal systems such as law, or if [it doesn't] apply because law is a non-formal system, then for that reason the Indeterminacy thesis is proven." D'Amato, *supra* note 284, at 176 n.92. Readers may want to digest this argument for themselves, but the following questions may be helpful as a starting point. (1) If Gödel's First Theorem doesn't apply, why might it not be the case that some other mathematics does apply, and the other mathematics suggests that law is "determinate"? After all, Gödel's First Theorem applies only to certain types of formal systems. (2) If mathematics doesn't apply, how does it follow that law is indeterminate?

The positions of Kornstein and Farago are cited with approval in Buckley, *supra* note 345, at 904 n.166. Buckley also asserts that Gödel's work shows that we "cannot fully describe the structure of law." *Id.* According to Buckley, this application of Gödel is justified because "law is self-referential" and "when we seek to define law, we do so in its own terms." *Id.* There is no elaboration on these references to the encoding features.

354. Brown & Greenberg, *supra* note 4, at 1479 n.181.

355. *See* Stone-de Montpensier, *supra* note 313; Roy L. Stone-de Montpensier, *The Complete Wrangler*, 50 Minn. L. Rev. 1001 (1966) [hereinafter Stone-de Montpensier, *Wrangler*]; Stone, *supra* note 320.

Some scholars place Stone-de Montpensier in the first group discussed—namely, those who try to satisfy the hypotheses of (something like) Gödel's Theorems. *See* Brown & Greenberg, *supra* note 4, at 1470 & n.161; Sinclair, *supra* note 352, at 363 n.33. Stone-de Montpensier does say that "[Gödel] seems to apply [to law]." Stone-de Montpensier, *Wrangler*, *supra*, at 1002; Stone, *supra* note 320, at 281. To my mind, however, the discussion in his writings, taken as a whole, suggests that Stone-de Montpensier is in the second group.

described above. In any case, what these scholars have done is simply attempt to find legal analogues of the ancient Liar-type Paradoxes.[356] The Greeks had legal versions of these paradoxes themselves.[357] Given the fact that these paradoxes are millennia old and extensively discussed,[358] it is not clear what is added to the specific legal debates by the invocation of Gödel's work outside of the observation that Gödel also relied on these same paradoxes.

Finally, others hope to draw lessons by analyzing the arguments that establish Gödel's First Theorem. One scholar, noting that the First Theorem does involve a certain balancing, concludes as follows:

> Gödel's result remains valid so far as formal modes of argument are concerned . . . but does not extend to paraductive arguments. This is why, in spite of appearances, the legal system may be complete . . . and must be consistent.[359]

But what is the miraculous cure of paraduction?

> Paraduction is a method of argument, *similia e similibus*, case by case, which is appropriate to a priori, nonnecessary connection.[360]

This appeal to some case-by-case, extra-formal, or intuition-based analysis is made by other commentators who believe that Gödel's work applies to law.[361] Indeed, scholars in general seem to believe that this is one of the lessons that the mathematical community has learned.[362] The fact, however, is that a great controversy exists in the mathematical community about whether the limitations that Gödel's Theorems places on certain types of formal systems apply to human reasoning as well. Indeed, one commentator opines that this issue "has generated more

356. In fact, they admit as much. Brown & Greenberg, *supra* note 4, at 1474; Stone-de Montpensier, *supra* note 313, at 669; Stone-de Montpensier, *Wrangler*, *supra* note 355, at 1015; Stone, *supra* note 320, at 281.

357. *See* J.C. Hicks, *The Liar Paradox in Legal Reasoning*, 29 Cambridge L.J. 275, 275-76 (1971).

358. For numerous articles on the Liar Paradoxes, see *The Paradox of the Liar* (Robert L. Martin ed., 1970); *Recent Essays on Truth and the Liar Paradox* (Robert L. Martin ed., 1984).

359. Stone-de Montpensier, *supra* note 313, at 670.

360. *Id.* at 671.

361. *See* Kornstein, *supra* note 9, at 127–28; Brown & Greenberg, *supra* note 4, at 1481, 1485; Farago, *supra* note 322, at 236–39.

362. *See* Brown & Greenberg, *supra* note 4, at 1468, 1487; Kaye, *supra* note 321, at 617 n.71; Stone-de Montpensier, *supra* note 313, at 670 & n.55.

discussion than any other . . . on the philosophical import of [Gödel's First Theorem]."[363]

For the academic legal scholar, perhaps one lesson comes from the realization that mathematics has grown, flourished, and renewed itself, because, not in spite, of its foundational crises.[364] This realization might, for example, lead to a different emphasis in the analysis of the growth of movements such as legal realism and critical legal studies.[365]

For the working lawyer, the lessons are even less clear. Quite frankly, many working mathematicians are not overly impressed by Gödel's specific results.[366] Gödel's First Theorem tells the working number theorist, for example, that given a formal system satisfying certain properties, there is a certain informal statement about the natural numbers whose formal counterpart cannot be proved or disproved within the system. Her first response will be, "Why should I care about this statement?" It will be no good to tell her that metatheoretically the statement can be viewed as saying something like, "I am unprovable in this system," for she will want to know whether the statement says anything interesting about the properties of natural numbers. In this regard, it is worth noting that very few such interesting statements have been found with respect to some of the standard formal systems of interest to number theorists. Whereas Gödel's First Theorem was proved in 1931, it was not until the late 1970s that mathematicians found an interesting arithmetic statement whose formal counterpart was undecidable with respect to the traditional formal system called Peano Arithmetic.[367] In addition, there is no known interesting arithmetic statement whose formal counterpart has been shown undecidable with respect to the more complicated set-theoretic systems in which number theorists routinely work.[368] Indeed, one legal commentator suggests that

363. Howard de Long, *A Profile of Mathematical Logic* 273 (1971). There is voluminous literature on this issue. *See id.* For brief introductions to the contours of the debate, see Michael A. Arbib, *Brains, Machines, and Mathematics* 138–40 (1964); Michael Barr, Book Review, 97 Am. Mathematical Monthly 938 (1990) (reviewing Penrose, *supra* note 344). Although not referring to this debate specifically, Rogers and Molzon do raise the issue. *See* Rogers & Molzon, *supra* note 4, at 1010 n.52.

364. Western thought has grown similarly.

365. *Cf.* Banner, *supra* note 45, at 253 n.33.

366. *See* King, *supra* note 32, at 54.

367. *See* Jeff Paris & Leo Harrington, *A Mathematical Incompleteness in Peano Arithmetic*, in *Handbook*, *supra* note 253, at 1133, 1133 n.*.

368. Telephone Interview with Peter Hinman, Professor of Mathematics, University of Michigan (Feb. 1996).

"[a]s in arithmetic, the important [legal] issues might be resolvable."[369] Having said this about arithmetic, it must be admitted that there are a number of interesting non-arithmetic sentences undecidable with respect to traditional set-theoretic systems.[370] Still, this fact has not led working set theorists to despair but to the consideration of potential new set-theoretic axioms and their implications.[371] As far as the Second Theorem is concerned, most working mathematicians have come to accept the validity of infinitary reasoning.[372] And as Hilbert intuited with respect to his conservation goal,[373] in many situations arithmetic sentences provable in more complicated systems can be derived in a simpler system whose set of axioms is augmented with the assumption that the more complicated system is consistent.[374] Moreover, consistency is an issue for working mathematicians only if an inconsistency arises. In each of the three crises, mathematicians were content to continue doing their work once they developed techniques that seemingly eliminated the specific inconsistencies confronting them. One might speculate that working lawyers in a common law system must be similarly charitable.[375]

2. *The Crisis and the Critique of Science*

As described in section D of part II, the past 150 years have seen a complex and comprehensive reevaluation of the Western intellectual tradition. In particular, a variety of intellectual currents have led scholars to question traditional approaches to what this Article refers to as science (i.e. classification).

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369. Saunders, *supra* note 314, at 229. *Cf.* Jacob, *supra* note 321, at 1657 n.126. On the other hand, D'Amato opines that "if we peruse the most recent 1,000 cases on free exercise of religion under the first amendment, . . . I would not be surprised if at the very least 999 of them are Godelian undecidables." D'Amato, *supra* note 284, at 173 n.80. *See also* Brown & Greenberg, *supra* note 4, at 1481.

370. As has been noted in connection with non-Euclidean geometry, Gödel's work does not represent the only machinery available for showing that certain sentences are undecidable with respect to certain formal systems. *See* text accompanying *supra* notes 220–21 For a detailed discussion of methods used in connection with set-theoretic systems, see Kenneth Kunen, *Set Theory: An Introduction to Independence Proofs* (1980). However, I do not know of any attempts to apply such techniques to law.

371. *See* J.R. Shoenfield, *Axioms of Set Theory*, in *Handbook*, *supra* note 253, at 321, 341–44.

372. *See* Michael J. Beeson, *Foundations of Constructive Mathematics* 431 (1985); *see also* Davis & Hersh, *supra* note 34, at 152–57.

373. *See supra* text accompanying note 269.

374. *See* Smorynski, *supra* note 253, at 858.

375. *See* Greenwood, *supra* note 237, at 576. For a similar assertion from an author steeped in the civil law tradition, see Ilmar Tammelo, *Modern Logic in the Service of Law* 127–28 (1978).

In describing the American incarnation of the overall reevaluation, Williams and Purcell single out the evolution of American pragmatism, the importation of logical positivism, and the impact of certain developments in mathematics and physics, especially non-Euclidean geometry, Gödel's Theorems, relativity, and quantum physics.[376] The intent here is not to evaluate the ultimate conclusions of the reevaluation, but to ask whether scholars have critically considered the content, context, and relevance of the mathematical material. Williams and Purcell have placed legal realist and critical legal studies scholarship in this larger intellectual context, and this explains how esoteric mathematical results have made their way into the legal literature.[377]

Purcell comments on the impact of non-Euclidean geometry on scholars in other fields as follows:

> The impact on mathematicians and geometers, who had always assumed that only one geometry was possible, was staggering. If Euclid's geometry was true, then the non-Euclidean geometries had in some way to be false. Yet no one was able to find any contradictions or inconsistencies . . . . Gradually during the latter half of the nineteenth century mathematicians and geometers came to understand that geometries were, in themselves, wholly formal systems with no necessary connection with any empirical reality. Geometries were logical systems, based on arbitrary postulates, whose only necessary characteristic was self-consistency.
>
> That discovery, although it implied a challenge to the belief in synthetic *a priori* knowledge, would perhaps not have had such a great impact had it not been for the work of . . . Einstein. . . . Among his monumental achievements, he demonstrated conclusively that Euclidean geometry did not completely describe the physical universe. . . .
>
> . . . .
>
> The popular discussion of non-Euclidean geometry was one of the broad results of the confirmation of the theory of relativity. . . . Axioms and principles were "free creations of the human mind,"

376. *See* Purcell, *supra* note 51, at 47–73; Williams, *supra* note 12; *see also* James Boyle, *The Politics of Reason: Critical Legal Theory and Local Social Thought*, 133 U. Pa. L. Rev. 685, 730–32 & n.141 (1985); D'Amato, *supra* note 284, at 152 n.16; Roberta Kevelson, *Semiotics and the Law*, 61 Ind. L.J. 355, 364–65 (1986); Levit, *supra* note 320, at 136–37 & n.3; McDougall, *supra* note 319, at 89; Stick, *supra* note 314, at 343.

377. *See* Purcell, *supra* note 51, at 62–63, 74–94; Williams, *supra* note 12.

Einstein insisted in 1922, and as such had necessarily "to be taken in a purely formal sense, i.e. as void of all content of intuition or experience." All deductive systems were formal creations that might or might not connect with the world of physical reality. "One geometry cannot be more true than another," declared the great French mathematician Henri Poincaré . . . . "[I]t can only be more convenient." Deduction could discover or prove nothing, and belief in synthetic *a priori* knowledge lacked any scientific, geometric, or logical basis. There was no reason to think that any allegedly self-evident truth was either self-evident or true. "The Kantians," summarized one scientist, "were most decidedly in the wrong when they assumed that the axioms of geometry constituted *a priori* synthetic judgements transcending reason and experience."

The discovery, popularization, and scientific utility of non-Euclidean geometry helped create a widespread belief in the non-Euclidean possibilities of all lines of reasoning. The characteristics of geometry, many argued, were clearly the characteristics of all fields of deductive thought; it followed that social thought, political theory, and ethics could all produce non-Euclidean or nonconventional systems that would be as valid logically as the most traditional and thoroughly accepted theories. . . . The concept of non- Euclideanism, generalized to include all types of deductive thought, robbed every rational system of any claim to be in any sense true, except insofar as it could be proved empirically to describe what actually existed.

. . . Non-Euclideanism came, in fact, to stand not only for the presumably unchallengeable logico-mathematical proof of the inherent formalism of deductive reasoning, but also metaphorically for almost any new or radical hypothesis that might be put forward.[378]

378. Purcell, *supra* note 51, at 50–53. This type of metaphorical use of non-Euclidean geometry is alive and well in modern legal scholarship. For examples, see Anthony V. Alfieri, *Impoverished Practices*, 81 Geo. L.J. 2567, 2610 n.190 (1993); Ronald J. Allen, *The Explanatory Value of Analyzing Codifications by Reference to Organizing Principles Other Than Those Employed in the Codification*, 79 Nw. U. L. Rev. 1080, 1081–82 (1985); Robert H. Bork, *The Place of Antitrust Among National Goals*, in Nat'l Indus. Conference Bd., *Basic Antitrust Questions in the Middle Sixties: Fifth Conference on the Impact of Antitrust on Economic Growth* 18 (1966); Ronald Cass, *Sentencing Corporations: The Guidelines' White Collar Blues*, 71 B.U. L. Rev. 291, 299 (1991); Jerome Frank, *Mr. Justice Holmes and Non-Euclidean Legal Thinking*, 17 Cornell L.Q. 568 (1931); Ronald Kahn, *The Supreme Court As a (Counter) Majoritarian Institution: Misperceptions of the Warren, Burger, and Rehnquist Courts*, 1994 Det. C.L. Rev. 1, 59; Ronald C. Kahn, *God Save Us from the Coercion Test: Constitutive Decisionmaking, Polity Principles, and Religious Freedom*, 43 Case W. Res. L. Rev. 983, 1019 n.164 (1993); Leiser, *supra* note 217, at 60; Frank I. Michelman,

Williams has a similar description of the impact of non-Euclidean geometry on what she describes as the "first wave" scholars:

> [Intellectuals] focused on the most obvious implications of non-Euclidean geometries—that abstract, deductive logic has no necessary connection with an external reality. Thus, they concluded that theories do not describe an objective reality, but that facts do.
>
> After considering the implications of non-Euclidean geometries, . . . intellectuals concluded that the only way to determine whether a given logical system has any connection with the real world is to test its predictions empirically. In the words of one contemporary mathematician, Henri Poincaré: "One geometry cannot be more true than another; it can only be more convenient." [As Purcell puts it,] [i]f one theory of mathematics proves to be empirically inaccurate, it is not because "the 'mathematics is wrong, but only [because] we have chosen the wrong mathematics!'"[379]

As Purcell and Williams describe the overall reevaluation, a typical invocation of the mathematical developments involves a two-step argument: (1) mathematics has been forced to reject some position, (2) hence so must some other field.

As far as the second step in the invocation is concerned, it is not clear why decisions within mathematics should be dispositive of issues in other disciplines any more than decisions in some other discipline should be dispositive of issues in mathematics. Mathematics is certainly a central concern *of* the Western intellectual tradition, but it is not *the* Western intellectual tradition. However appealing as a psychological matter, the second step requires a justification. Without one, such an implication ignores the question of the relevance and separateness of the other discipline.[380]

The Parts and the Whole: Non-Euclidean Curricular Geometry, 32 J. Legal Educ. 352 (1982); Peritz, *supra* note 217, at 1251–52; Stone, *supra* note 321, at 74.

379. Williams, *supra* note 12, at 441.

380. With respect to this second step, one might want to consider the following:

> Constitutional law is not mathematics—but one must wonder why, if mathematicians in this post-Gödelian age treat as inevitable the fact that interesting logical systems are open-ended, constitutional lawyers continue to demand that their universe of discourse be closed.

Lawrence H. Tribe, *American Constitutional Law* 47 (1978).

Imagine, however, seeing the following hypothetical quotation in a 1978 mathematics journal:

> Mathematics is not constitutional law—but one must wonder why if constitutional scholars in this post-Warren Age demand that their universe of discourse be closed, mathematicians

More troubling is the first step of the typical invocation. Scholars in other fields betray a lack of understanding of the content and context of the mathematical material itself. The mathematical community simply has not reached the conclusion that non-Euclidean geometry forecloses any particular scientific stance. The reader should be aware that not all scholars even accept the conclusion that non-Euclidean geometry destroys Kant's geometric approach. Körner puts it as follows:

> The distinction which Kant makes . . . between the thought of a mathematical concept, which requires merely internal consistency, and its construction, which requires that perceptual space should have a certain structure, is most important for the understanding of his philosophy. Kant does *not* deny the possibility of self-consistent geometries other than the ordinary Euclidean; and in this respect he has not been refuted by the actual development of such geometries.[381]

Moreover, even if the Kantian conception of geometry is destroyed, it does not follow that all Kantian conceptions of mathematics are untenable. In this regard, the reliance on Poincaré is interesting. Poincaré's original words are as follows:

> If geometry were an experimental science, it would not be an exact science. It would be subjected to continual revision. . . . *The geometrical axioms are therefore neither synthetic à priori intuitions nor experimental facts.* They are conventions. Our choice among all possible conventions is *guided* by experimental facts; but it remains *free*, and is only limited by the necessity of avoiding every contradiction, and thus it is that postulates may remain rigorously true even when the experimental laws which have determined their adoption are only approximate. In other words, *the axioms of geometry* (I do not speak of those of arithmetic) *are only definitions in disguise.* What, then, are we to think of the question: Is Euclidean geometry true? It has no meaning. We might as well ask if the metric system is true, and if the old weights and measures are false; if Cartesian co-ordinates are true and polar co-ordinates

continue to assert that logical systems of the sort to which Gödel's First Theorem applies are interesting.

381. Körner, *supra* note 97, at 28–29. *See also* J. Alberto Coffa, *The Semantic Tradition from Kant to Carnap* 189 (1991).

> false. One geometry cannot be more true than another; it can only be more convenient.[382]

As the more complete quotation indicates, Poincaré specifically exempts arithmetic from his discussion. Indeed, Poincaré believed in a Kantian type of intuition about arithmetic on which the whole of mathematics is based.[383] According to Leopold Kronecker, one of Poincaré's intellectual siblings and one of the forerunners of Intuitionism,[384] "God made integers, all else is the work of man."[385] The Intuitionists in general accepted much of Kant outside of his geometric stance.[386] In any case, as the quotation indicates, Poincaré is advancing a geometric conventionalism not a geometric empiricism. Generally speaking, geometric conventionalism asserts that experiments cannot distinguish between geometries since one can posit compensating forces affecting measuring instruments. Therefore the choice of a "real world" geometry is merely conventional, not empirical.[387]

In fact, it is not at all clear that non-Euclidean geometry forecloses any particular stance, let alone Kant's. One need only consider the wide range of positions held by and within the three schools that emerged at the turn of the century (hence after the development of non-Euclidean geometry).[388]

Similar comments on the first and second steps can be made with respect to the invocations of Gödel. With respect to the first step, Purcell comments on the invocation of Gödel as follows:

> Any system of mathematics or logic was thus based on certain axioms and postulates assumed at the start. There was no need to prove them since foundation postulates were necessary and unavoidable. There was no way to prove them, either, since initial postulates were wholly arbitrary. From those initial assumptions the mathematician or logician developed his system with no thought of its descriptive or empirical applicability. His only criterion was internal consistency. "In pure geometry what is demonstrated,"

382. Henri Poincaré, *Science and Hypothesis* 49–50 (1905).

383. *See* Fraenkel et al., *supra* note 3, at 253; *see also* Max Black, *The Nature of Mathematics: A Critical Survey* 178 (1933).

384. *See* Fraenkel et al., *supra* note 3, at 253.

385. *Quoted in* Robert E. Moritz, *On Mathematics* 269 (1942).

386. *See* Körner, *supra* note 97, at 119–55.

387. For an overview of geometric conventionalism, see Lawrence Sklar, *Space, Time, and Spacetime* 85–146 (1974).

388. *See supra* text accompanying notes 160–270.

> declared the positivist philosopher Albert E. Blumberg, "is a theorem or more precisely the relation of analytic deducibility or tautological implication between postulate-set and theorem." All logico-mathematical reasoning was thus purely tautological, the elaboration of implications contained by definition in the foundation postulates. In 1930, Kurt Gödel, one of the logical positivists who came to the United States, produced his "incompleteness theorems," with which he demonstrated to the satisfaction of most of his colleagues that it was theoretically impossible to produce any final or ultimate solution to the problem of the foundations of mathematical logic. All of the competing schools of mathematics, claimed E.R. Hedrick, the chairman of the mathematics section of the American Association for the Advancement of Science in 1932, accepted the full implications of non-Euclidean geometry and the formal nature of mathematical reasoning. "I may assume that no one of these schools would attempt to base its system on a claim of reality."[389]

With respect to the second step, Williams describes Gödel's effect on what she calls the "second wave" scholars as follows:

> [T]he mathematician Kurt Gödel's "incompleteness theorem" played an important metaphorical role in the thinking of intellectuals outside of math and physics. Gödel's mathematical proof demonstrated that arithmetic cannot be both complete and internally consistent. The incompleteness theorem . . . reinforced the conviction of second-wave scholars that languages, including mathematics, are necessarily incomplete descriptions of reality.[390]

Williams further elaborates as follows:

> [S]cholars began to argue that neither scientific nor nonscientific disciplines could gain access to objective truth, but instead could only provide interpretations of "texts."[391]

It is wrong to describe Gödel as a logical positivist as Purcell does. Gödel himself said, "I never was a logical pos[itivist]."[392] Although Gödel attended meetings of the Vienna Circle, he gradually moved away

389. Purcell, *supra* note 51, at 55–56.

390. Williams, *supra* note 12, at 439.

391. *Id.* at 454.

392. *Quoted in* Wang, *supra* note 247, at 49.

from it. In any case, he never did agree with their tenet that mathematical truth was a convention of language.[393]

Gödel's foundational approach used set theory.[394] The set-theoretic approach treats sets as fundamental. It can be traced to Cantor's work with sets and is in some ways the successor to the early Logicist reduction program of the *Principia*. As a foundational approach, serious problems stem from the development, reminiscent of what took place in geometry, of a whole collection of equally consistent set theories. As with Logicism, Formalism, and Intuitionism, the set-theoretic approach encompasses a wide variety of philosophical positions.

In any case, as E.R. Hedrick and second-wave scholars should note, Gödel was a Platonist who rejected skepticism![395] In Gödel's words:

> [T]he assumption of [set-theoretic objects] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perception . . . .[396]

Gödel also says:

> [L]ogic and mathematics (just as physics) are built up on axioms with a real content which cannot be "explained away."[397]

Gödel's epistemological position had both rationalist and consequentialist strands:

> [E]ven disregarding the intrinsic necessity of some new [set-theoretic axiom], and even in case it has no intrinsic necessity at all, a probable decision about its truth is possible also in another way, namely, inductively by studying its "success." Success here means fruitfulness in consequences, in particular in "verifiable" consequences, i.e. consequences demonstrable without the new axiom, whose proofs with the help of the new axiom, however, are considerably simpler and easier to discover, and make it possible to contract into one proof many different proofs. . . . A much higher

393. *See* Feferman, *supra* note 305, at 101.

394. For general overviews of set-theoretic approaches, see Maddy, *supra* note 167; Steven Pollard, *Philosophical Introduction to Set Theory* (1990); Tiles, *supra* note 116.

395. Stick alludes to Gödel's stance without detail. *See* Stick, *supra* note 314, at 398 & n.282.

396. Kurt Gödel, *Russell's Mathematical Logic*, *in* Benacerraf & Putnam, *supra* note 161, at 447, 456–57.

397. *Id.* at 461.

> degree of verification than that, however, is conceivable. There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems . . . that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory.[398]

Gödel's philosophical perspective was critical to the development of his mathematical work, *including the incompleteness theorems*![399] In Gödel's own words:

> [M]y objectivist conception of mathematics . . . in general, and of [infinitary] reasoning in particular, was fundamental . . . .
>
> How indeed could one think of *expressing* [statements about mathematics] *in* the mathematical systems themselves, if the latter are considered to consist of meaningless symbols . . . ?
>
> . . . .
>
> . . . [I]t should be noted that the heuristic principle of my construction [of the Gödel statement] is the . . . concept of . . . "mathematical truth", as *opposed* to that of [provability].[400]

Given Gödel's own view and the detailed contextual presentation of his results in the preceding Section, one might want to consider carefully the vague but sweeping assertions that have appeared in the legal literature such as, "[I]t should not be surprising to find that the philosophical implication of Gödel's theorem should question the basic premise of philosophy—that is, the basic question of whether reality exists."[401] With respect to such assertions, Abraham Fraenkel, Yehoshua Bar-Hillel, and Azriel Levy say the following:

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398. Kurt Gödel, *What Is Cantor's Continuum Problem?*, *in* Benacerraf & Putnam, *supra* note 161, at 477. For a discussion of this position, see Maddy, *supra* note 167, at 31–35.

399. *See* Feferman, *supra* note 305, at 106–08.

400. *Quoted in* Wang, *supra* note 247, at 9.

401. Kelso, *supra* note 11, at 833 n.44. *See also id.* at 833 n.46 ("Father Kung concludes, as Gödel's theorem implies, that not only the decision to believe in God but also the decision to believe in reality is one ultimately based on trust.").

For another example, consider the following statement by Donald Gjerdingen:

> By Gödelian worlds, I simply mean the general idea, borrowed from Kurt Gödel's work in math theory, that (a) we construct our worlds—both theoretical and real—out of nested systems of thought that fold into each other; (b) each system is based on certain axiomatic assumptions that cannot be proved within the system they create; (c) multiple worlds, each consistent in its own

> way, can be constructed; and (d) such systems are self-referencing. . . . I believe that such ideas can be applied on a far grander scale and in far different ways than lawyers have yet been willing to talk about.

Donald H. Gjerdingen, *The Future of Our Past: The Legal Mind and the Legacy of Classical Common-Law Thought*, 68 Ind. L.J. 743, 748 n.3 (1993) (citations omitted).

Peritz says the following:

> Some sixty years ago, mathematician and logician Kurt Gödel published his Incompleteness Theorem . . . a project whose importance the community of theoretical mathematicians and philosophers did not recognize for many years. . . . Gödel . . . exposed the distressing limitations of Bertrand Russell's neo-Platonic vision of mathematics as the only pure deal for all of our less precise and thus less aesthetically pleasing practices—philosophy or law, for example. Russell's idealism crumbled under the weight of Gödel's Theorem. The world of numbers and sets turns out to be less than perfectly predictable. Even the purest abstraction cannot provide the vehicle for returning to a philosophical or empirical Eden—to the nominalist's Garden of stability and control. Yet most of us embrace science as having falsified Nietzsche's unnerving claim that final authority for any proposition is ultimately unavailable—that God is dead.

Peritz, *supra* note 320, at 999 n.214.

Terrell says the following:

> Another way to state this sense of perceptual incompleteness is to argue that our logical capabilities as a whole are a kind of tautology—that is a system of rules with internal consistency that nevertheless cannot produce a verification or justification of itself. The best and most basic example of a tautological system is mathematics, which is a subject of our logic. Its lack of an internal verification is captured in Gödel's Theorem . . . .

Terrell, *supra* note 24, at 319 n.92.

For other examples of broad claims, see Levit, *supra* note 320, at 137 n.3 (noting "existence of indeterminacy in any explanatory system"); Loevinger, *supra* note 5, at 343 ("[A]s Kurt Gödel demonstrated, any formal logical system ultimately rests on some undecidable—that is, unprovable—propositions. Thus there is some degree of uncertainty in all proofs—scientific, legal, philosophical, social, or intuitive.") (footnote omitted); Scanlan, *supra* note 318, at 1525 (invoking Gödel's work as support for proposition that "[a] characteristic of much of modern thought extending across a broad spectrum of disciplines is to deny that we all experience the same things, speak a common language, or employ a single coherent logic in addressing our various concerns"); Spann, *Secret*, *supra* note 314, at 698 n.58 (stating that one of implications of Gödel's work is that logic is not "a closed system in which things happen in a predictable way, in accordance with orderly rules that are understandable and reliable"); Tom Stacy, *Death, Privacy, and the Free Exercise of Religion*, 77 Cornell L. Rev. 490, 568 (1992) (citing Gödel's work in support of assertion that contemporary thinkers place emphasis on "limitations of reason"); Stephan, *supra* note 312, at 750–51 & n.5 ("[T]here are good reasons to handle the concept of rationality with caution: at least since Goedel proposed his eponymous theorem, there has been good reason to believe that mathematics, supposedly the purest expression of human reason, rests at bottom on begged questions rather than logical proof."); Yudof, *supra* note 335, at 690 ("There are only degrees of certainty and consistency, even in mathematics."); Greenwood, *supra* note 237, at 576 (using Gödel's work as support for argument that inconsistency does not necessarily cause trouble for moral or legal stance).

For another example, see Stevens v. Tillman, 855 F.2d 394, 399 (7th Cir. 1988) (citing Gödel's work as support for proposition that "[c]ourts trying to find one formula to separate 'fact' from 'opinion' . . . are engaged in a snipe hunt"), *cert. denied*, 489 U.S. 1065 (1989).

One also might ask what possibly could be meant by Jeanne L. Schroeder, *Abduction from the Seraglio: Feminist Methodologies and the Logic of Imagination*, 70 Tex. L. Rev. 109, 130 & n.47 (1991) ("If relationalism seeks to reconcile the self and the other, while respecting the otherness of the other, it may be intensely feminist. But is it characteristically feminine? Relationalism

> Many attempts have been made to interpret . . . Gödel's incompleteness theorem as discrediting certain ontological views and bolstering others. We do not believe that these attempts were successful.[402]

Even in the wake of non-Euclidean geometry and Gödel's Theorems, the philosophy of mathematics has continued to encompass wide-ranging foundational stances. Various strains of Logicism, Intuitionism, and Formalism have evolved, and a number of new approaches have developed.[403] Also, as the quotation at the beginning of this part indicates, Platonism is alive and well in mathematics.[404] Indeed, Philip Davis and Reuben Hersh write:

> Most writers on the subject seem to agree that typical working mathematician is a Platonist on weekdays and a formalist on Sundays. That is, when he is doing mathematics he is convinced that he is dealing with an objective reality whose properties he is attempting to determine. But then, when challenged to give a philosophical account of this reality, he finds it easiest *to pretend* that he does not believe in it after all.[405]

The assertion that the rejection of objective truth and certainty has "permeated . . . mathematics"[406] is misleading, if not inaccurate.

One might begin to wonder about the interdisciplinary payoff of efforts to use the current foundational crisis.[407] On the one hand, it is interesting that important issues involving the concept of self-reference appear in disciplines as diverse as mathematics and law.[408] Moreover, the legal perspective provided by an analysis of legal analogues of the Liar-type Paradoxes no doubt will help shed light on this multi-faceted concept. On the other hand, the tendency to exalt the concept over its particular disciplinary manifestations has led to what one commentator

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characterizes much contemporary scientific theory . . . [including] Gödel's theory of mathematical systems . . . .").

402. Fraenkel et al., *supra* note 3, at 342.

403. For brief looks at the diversity of views, see Beeson, *supra* note 372, at xiii-xvii, 417–38; Fraenkel et al., *supra* note 3, at 331–45; Maddy, *supra* note 167, at 1–36; Benacerraf & Putnam, *supra* note 161, at 1; Kitcher & Aspray, *supra* note 98; *supra* note 317.

404. *See supra* text accompanying note 93.

405. Davis & Hersh, *supra* note 34, at 321 (emphasis added).

406. Williams, *supra* note 12, at 430–31.

407. *See* Sinclair, *supra* note 4, at 33 n.13; Jacob, *supra* note 321, at 1657 n.126; *see also* Dow, *supra* note 4, at 723–26; Kress, *Preface*, *supra* note 348, at 143–45.

408. There is much more to self reference in mathematics than is described in this Article.

describes as the "self-reference craze."[409] Such a tendency can lead, for example, to the use of Gödel's Theorem as a cultural metaphor for intellectual skepticism.[410] Not only is such a use a mischaracterization,[411] but the acceptance of vague and imprecise metaphorical reasoning in aid of claims of intellectual skepticism is somewhat circular.

V. CONCLUSION

> *[F]or two thousand five hundred years mathematicians have been correcting their errors to the consequent enrichment and not impoverishment of their [discipline]; and this gives them the right to face the future with serenity.*
>
> *Nicholas Bourbaki*[412]

It is commendable to look to other disciplines when grappling with difficult issues in one's own field, but care must be taken to appreciate both the content and context of other work. The discussion of foundational concerns in mathematics provides a sobering illustration.

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409. *See* Noam Cohen, *Meta-Musings: The Self-Reference Craze*, The New Republic, Sept. 5, 1988, at 17; *see also* Sinclair, *supra* note 4, at 33 n.13.

410. *See supra* text accompanying notes 390–91; *supra* note 401 and accompanying text. For another example of Gödel as a metaphor, consider Pierre Schlag:

> Thus, when Fiss tried to fend off nihilism by attempting to constrain the interpretation of legal rules (with more and better "disciplining rules"), Fish pulled out the infinite regress. A "disciplining rule" is still a rule. And thus Fiss' "disciplining rules" have all the same problems as the ordinary low level legal rules. (If all texts are indeterminate, then it's a pretty good bet (if you believe in Gödel) that they can't be shown to be determinate with more text.)

Schlag, *supra* note 335, at 40 (footnotes omitted). Schlag continues:

> For metaphorical analogues of the [Gödel's First Theorem] in the legal context, see [David Kennedy's argument that] modern legal thought masks conflict through indeterminacy[, and Schlag's arguments that] legal distinctions become self-destructive [and] conventional ways of understanding rules vs. standards debate only replicate this dispute[.]

Id. at 40 n.16 (citations omitted).

Boris Bittker uses Gödel to describe the inability of economists to make precise predictions. *See* Bittker, *supra* note 310, at 220; *see also* Banner, *supra* note 45, at 253 & n.33 (citing Gödel as "analogy" to statement that "[t]o the extent that a system aspires to completeness, to including every person and every life situation, it sacrifices impartiality, the ability to find someone or something outside the system to serve as a meta-authority"); Spann, *Secret*, *supra* note 314, at 698 n.58 (using Gödel as metaphor for uneasiness about ability of legal system to answer questions in a determinate manner).

411. *See supra* notes 389–406 and accompanying text.

412. Nicholas Bourbaki, *Elements of Mathematics: Theory of Sets* 13 (1968). Nicholas Bourbaki is actually the pen name for a group of French mathematicians.

Descriptions of the mathematics are misleading at best. In addition, many scholars use unjustified chains of reasoning to make sweeping conclusions. Virtually all scholars ignore the fact that the so-called current foundational crisis is merely one in a sequence dealing with the same basic questions, and that these crises are symptomatic of larger periodic internal reevaluations of the Western intellectual tradition. The mathematical crises have not so much resolved as sharpened understanding of the basic mathematical issues involved. As a result of its crises, mathematics has matured scientifically, artistically, technologically. One can only hope that other disciplines such as law are as successful in using their opportunities.[413]

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413. *Cf.* Curran, *supra* note 321, at 28 n.71 (discussing views of Jonathan Culler and George Steiner); George P. Fletcher, *Paradoxes in Legal Thought*, 85 Colum. L. Rev. 1263 (1985).

Many of the major points made in this Article are highlighted by Anthony D'Amato's invocation of the so-called Löwenheim-Skolem Theorems.

One point is that providing a specific context for complex mathematical results is as important to understanding them as a description of the work itself. The context provided here for the Löwenheim-Skolem Theorems involves various strands of Logicist and Formalist ideas.

Recall that one of the developments leading to Logicism was the search for a symbolic notation for the laws of logic. *See supra* text accompanying notes 162–63. George Boole's system could describe the elementary logic of classes. That is, it could symbolize such phrases as "the complement of the class of people over twenty five." But logicians wanted to be able to symbolize statements such as, "There is a person who is a friend of a person over twenty five." They wanted a language that could treat both (1) quantification ("there is"), and (2) general n-place relations ("is a friend of"–a two place relation) not just one-place relations ("over twenty-five"). *See* Kneale & Kneale, *supra* note 163, at 404–34. Thus the language should consist of (1) variables for individuals $x_1, x_2, \ldots$; (2) the propositional connective symbols; (3) the quantifiers $\exists$ and $\forall$; (4) the relation symbols $R_{(1,1)}$, $R_{(1,2)}, \ldots, R_{(2,1)}, R_{(2,2)}, \ldots$, where $R_{(i,j)}$ is the jth i-place relation symbol and one of the two-place relation symbols is singled out as the "equality" symbol; and (5) punctuation symbols. The formulas should be specified by (1) if R is an n place relation symbol and $x_{i1}, x_{i2}, \ldots, x_{in}$ are any n variables, then $R(x_{i1}, x_{i2}, \ldots, x_{in})$ is a formula; (2) if α and β are any two formulas then so are $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\neg\alpha)$, $(\alpha \rightarrow \beta)$, and $(\alpha \leftrightarrow \beta)$; and (3) if α is any formula and x_i is any variable, then $(\exists x_i\alpha)$ and $(\forall x_i\alpha)$ are formulas. Such a system is a so-called first-order system in that quantification is over variables not relation symbols. One might think that such a language is limited because it does not explicitly encompass function and constant symbols, but functions and constants are easily accommodated in formal systems using this language. *See* Davis, *supra* note 247, at 133 n.†.

Recall the idea of a sentence—a formula with no free variables. *See supra* note 278. For the system described in this footnote, any variable occurring in a formula of type (1) is free. A variable is free in a formula of type (2) if it is free in α or β. A variable is free in a formula of type (3) if it is free in α and is not x_i. We will call a set of sentences in this language a set of first-order sentences. A model for a collection of first-order sentences is a set A together with an "interpretation" under which the sentences are all true. What is an "interpretation"? Symbols other than relation symbols (e.g. propositional connectives) are given their intended meanings. Thus, an interpretation is essentially the assignment of an n-place relation on A to each of the n-place relation symbols appearing in the set of sentences, subject only to the requirement that the equality relation be assigned to the equality symbol. The size of the model is the size of its set.

In a 1915 paper, Löwenheim considered what models collections of first-order sentences could have. Leopold Löwenheim, *On Possibilities in the Calculus of Relatives*, in van Heijenoort, *supra* note 164, at 232. Why was he interested in this? One possibility is that he believed that his results might be of use in determining whether various axiom systems contain extraneous axioms. *Id.* at 240. This so-called independence issue was a concern of Hilbert's. *See* David Hilbert, *Foundations of Geometry* 32 (Paul Bernays ed., Leo Unger trans., 10th ed. 1971).

Whatever his motivations, Löwenheim's result was the first in a complicated web of results known as the Löwenheim-Skolem Theorems. Löwenheim's original result was extended in 1920 and 1922 by Thoralf Skolem, whose work in turn was generalized in one direction in 1936 by Anatolii Malcev to what is now called the Löwenheim-Skolem Downward Theorem, and in another direction in the late 1920s by Alfred Tarski to what is now called the Löwenheim-Skolem Upward Theorem. *See* Moore, *supra* note 233, at 251–58. These two theorems, in their various versions, comprise the basic Löwenheim-Skolem results and are sometimes combined into one proposition called the Löwenheim-Skolem Theorem. *See* J.L. Bell & A.B. Slomson, *Models and Ultraproducts: An Introduction* 82 (2d rev. ed. 1971). They were generalized further around 1960. *See id.* at 84–86 (describing so-called Löwenheim and Hanf numbers); Keith J. Devlin, *Constructibility* 332–33 (1984) (describing so-called Cardinal Transfer Theorems). Further complications stem from the fact that there are versions of each of the Upward and Downward Theorems equivalent to the so-called Axiom of Choice, perhaps the most controversial part of infinitary reasoning. *See* Moore, *supra* note 233, at 1, 258. Thus, some care must be taken to determine exactly what result is being described.

For the purposes of this Article, we present one version of each of the Downward and Upward Theorems. Skolem established the following version of the Löwenheim-Skolem Downward Theorem: If a set of first-order sentences has some model, then it has a model whose set has size at most that of the set of natural numbers. *See* van Heijenoort, *supra* note 164, at 228–32, 252–54 (describing history of Theorem). Why use the word "downward"? The idea is that even if the set of sentences has a model of "large" infinite size, one can find a model of "small" size (at most the infinity of the natural numbers, which is the "smallest" infinity). One version of the Löwenheim-Skolem Upward Theorem says that if a set of first-order sentences has a model whose size is that of the natural numbers, then it has models of all other infinite sizes as well. *See, e.g.*, Moore, *supra* note 233, at 257–58.

How do D'Amato and others do in describing these results? In one paper, D'Amato does quote Willard Quine's description of Skolem's result, although not in Quine's full context. *See* Anthony D'Amato, *Counterintuitive Consequences of "Plain Meaning"*, 33 Ariz. L. Rev. 529, 572 n.135 (1991). However, some of his other descriptions are vague. At one point, he says, "[T]he Löwenheim-Skolem theory in mathematics proved that, as to any given mathematical facts, an indefinite number of different theories can be constructed that are consistent with, and explanatory of, all the data." Anthony D'Amato, Letter from Anthony D'Amato to Editor-in-Chief, in 80 Am. J. Int'l L. 148, 149 (1986) (replying to letter written by Michael Akehurst). Other descriptions are incomplete at best. In one article, he says that "[w]hat is now known as the Löwenheim-Skolem theorem states that for any axiom system one may choose to characterize any mathematical set (e.g., the positive whole numbers), there is an infinite number of other interpretations that are drastically different and yet also satisfy the axiom system." D'Amato, *supra* note 321, at 599 n.102. In another article, he says the following:

> Leopold Löwenheim and Thoralf Skolem published a series of papers in the early 1920's and generally proved that, for any set of axioms that one chooses for characterizing any branch of mathematics, an infinite number of other interpretations are available that are drastically different and yet satisfy the chosen axioms as well.

Anthony D'Amato, *Can Any Legal Theory Constrain Any Judicial Decision?*, 43 U. Miami L. Rev. 513, 521 n.28 (1989) [hereinafter D'Amato, *Constrain*].

Quite frankly, it is not clear what result he is describing in the last two quotations. One must be very careful when using the word "characterize" and the phrase "drastically different." What is at issue in the results described above is the ability of sets of sentences to characterize *the size of their infinite models*. In this regard, the reader should keep the following four points in mind.

(1) There are, for example, sets of first-order sentences with models of all infinite sizes, hence unable to characterize the size of their infinite models, but all models of a given size are isomorphic (roughly speaking, there is a one-to-one correspondence preserving all the relations). In this sense, models of any given size are characterized. *See* Bell & Slomson, *supra*, at 178. (More technically, two models of a collection of sentences are isomorphic if the elements of the sets of the two models can be placed in one-to-one correspondence such that if *R* is any relation symbol appearing in the sentences, then its interpretation in the first model holds of any elements in the first model if and only if its interpretaion in the second model holds of the corresponding elements in the second model.)

(2) It is possible for a set of first-order sentences to characterize certain parts of mathematics dealing with sets of a fixed finite size in the sense that all models will have the same size—a finite size—and they will be isomorphic. *See* Bridge, *supra* note 324, at 103.

(3) Syntactically complete sets of first-order sentences have the property that all models, although possibly of different sizes, satisfy the same set of sentences. In this sense, the models are very difficult to distinguish. *See id.* at 100–01.

(4) Finally, one must be very careful when leaving the situation described in this footnote. On the one hand, for example, if the language embraces quantification over relations, then one can characterize the natural numbers up to isomorphism. *See* George Boolos & Richard Jeffrey, *Computability and Logic* 197–206 (2d ed. 1980); *cf.* Brown & Greenberg, *supra* note 4, at 1484 n.204; Kress, *Preface*, *supra* note 348, at 144. In this sense, the Löwenheim-Skolem Upward Theorem as described above does not hold. One also can provide counter examples to the Downward Theorem as described above. *See* Boolos & Jeffrey, *supra*, at 197–206. On the other hand, certain generalizations of the Löwenheim-Skolem results apply to very general situations, including situations involving languages embracing quantification over relations. *See* Bell & Slomson, *supra*, at 84–86 (discussing Löwenheim and Hanf numbers). In this sense, versions of the Löwenheim-Skolem Theorems do hold. With regard to the latter point, one must be very careful when considering statements such as "the Löwenheim-Skolem theorem . . . does not hold in second-order languages, which permit such quantifications," Kress, *Preface*, *supra* note 348, at 144, or "the proofs of Löwenheim and Skolem hold only for first-order formal systems," Brown & Greenberg, *supra* note 4, at 1484.

One also must consider carefully the content and context of the mathematics in applying the result specifically to law. D'Amato asserts that the hypotheses of something like the Löwenheim-Skolem Theorems are satisfied in law. As with his invocation of Gödel's Theorem, he does not attempt to establish this directly. *See* Kress, *Preface*, *supra* note 348, at 145. Rather, D'Amato appeals to others such as Raymond Smullyan and Stephan Körner, *see* D'Amato, *Constrain*, *supra*, at 521 n.28, and Morris Kline, *see* D'Amato, *supra* note 321, at 599 n.102. I have read the indicated citations and see nothing there to support his use of them. In addition, D'Amato makes the "damned if you do damned if you don't argument," analyzed *supra* note 353. *See* D'Amato, *supra* note 284, at 176 n.92. He also provides an arguably non-legal "example" of the theorem—namely, specifying only the first few numbers in a sequence does not determine the remaining numbers. *See id.* at 174 n.87; D'Amato, *supra* note 321, at 597 n.96. Clearly, this can be established without the Löwenheim-Skolem Theorems. If it is sufficient to make his point, then it is not clear what is added by the invocation of the Löwenheim-Skolem Theorems.

Finally, one should understand the implications that have been drawn in mathematics itself before drawing conclusions at large. D'Amato asserts that "the result reached by Lowenheim-Skolem . . . simply stated, is that ontology is indifferent to any formal system." D'Amato, *supra* note 284, at 175. Aside from the discussion in connection with (1)–(4) above, Abraham Fraenkel, Yehoshua Bar-Hillel, and Azriel Levy may be helpful: "Many attempts have been made to interpret . . . the Löwenheim-Skolem . . . theorem as discrediting certain ontological views and bolstering others. We do not believe that these attempts were successful." Fraenkel et al., *supra* note 3, at 342.